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Summary

Perturbation techniques based on Lie transforms as suggested by Deprit were used as the theoretical foundation for programming the analytical solution of the Main Problem in Satellite Theory (all gravitational harmonics being zero except J_2). The collection of formulas necessary and sufficient to construct an ephemeris is given in the exposition. Short and long period displacements, as well as the secular terms, have been obtained up to the third order in J_2 as power series of the eccentricity. They result from two successive completely canonical transformations which it has been found convenient not to compose into a unique transformation. Division by the eccentricity appears nowhere in the developments--neither explicitly nor implicitly. The determination of the constants of motion from the initial conditions has been given an elementary solution that is both complete and explicit without being iterative. The program was developed by Rom from MAO's package of subroutines for <u>M</u>echanized <u>A</u>lgebraic <u>O</u>perations. Reliability tests have been run in two instances: in-track errors for ANNA 1B are only 20 cm. after 210 days in orbit, while for RELAY II, they are 2.4 m. even after 350 days in orbit.

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Introduction

By the present communication, we announce that the theory of an artificial satellite can be produced explicitly in fully analytical form by means of programs which enable computers to process literal expressions. We justify this effort by displaying in two particular instances the accuracy yielded by the series over very long arcs.

Theories of artificial satellites can be characterized by the basic coordinates they use to map the phase space. For no other reason than our personal liking for Delaunay's elements, we have chosen to develop the Main Problem as set up by Brouwer (1959). This selection does not imply a judgment on the relative merits of Delaunay's variables versus spheroidal coordinates (Vinti, Kislik, Aksenov), elliptic elements derived from spheroidal coordinates (Iszak) or secularly processing elliptic elements (Sterne, Garfinkel, Aksnes). But we submit that comparison of analytical theories will not lead to definitive conclusions less we have the capabilities of generating each of them automatically by computer so that we can transfer analytically the constants of one theory into those of any other.

Our treatment of the Main Problem is original on five points:

(i) We discarded the so-called Von Zeipel's method, which is an algorithm devised by Poincaré (1893) to generate a canonical transformation from a determining function in mixed variables (old coordinates and new momenta). Instead, we use a formalism proposed elsewhere (Deprit 1969) under the name of Lie transforms. The advantage is that we generate explicitly the canonical transformations and their inverses without inversions or substitutions, and that we avail ourselves of a systematic procedure for transforming any state function into the new phase variables.

(ii) We gave up Brouwer's plan for a theory in a closed form. Indeed, although we could reproduce by Lie transforms the first order terms computed by Brouwer with Von Zeipel's method, we found that, at the second order, the quadrature for W_2 prescribed by the Lie transforms bears, among others, on terms of the type

$$
\begin{array}{ll}
(\ell-f)r^{-5}\sin 2g, & (\ell-f)r^{-4}\sin(f+2g) \\
(\ell-f)r^{-4}\sin 2g, & (\ell-f)r^{-4}\sin(f-2g) \\
(\ell-f)r^{-3}\sin 2g, & (\ell-f)r^{-3}\sin(f+2g) \\
 & (\ell-f)r^{-3}\sin(f-2g).
\end{array}
$$

We tried repeatedly to express in closed form the integrals of these functions over the mean anomaly. Actually, Moses (1969) indicated that such quadratures might not be expressible in closed form by means of the usual elementary functions. Similar terms have been encountered by Aksnes (1966) in his theory of an artificial satellite. In sum, the possibility of implementing Brouwer's theory beyond the second order in a closed form by means of rational, sine and cosine functions may now be regarded as an open question that might likely be answered in the negative. For this reason, we decided to expand the perturbing function in powers of the eccentricity.

(iii) Delaunay's mapping from polar coordinates to elliptic elements has the zero eccentricity as a singularity. It causes troubles in the corrections to be computed for the mean anomaly and the argument of perigee. One way of circumventing them would be to coordinatize the phase space by means of *eccentric* elements as defined by Poincaré or Hori. But it would generate considerable, although not insuperable, complications in the perturbation algorithm (see, for instance, Meffroy 1968). Instead, taking advantage of the systematic rule offered by Lie transforms for transposing state functions, we decided, on one hand, to retain Delaunay's elements as the phase coordinates, while, on the other hand, we base the ephemeris of the satellite on functions of these phase coordinates that are exempt from singularities for zero eccentricities. We have selected the mean distance $F = \ell + g$ to the node, the eccentric functions $C = e \cos g$, $S = e \sin g$, and the usual Delaunay's elements, namely the longitude h of the ascending node, the polar component H of the angular momentum, and the action $L = \sqrt{\mu a}$.

(iv) We obtain the short and long period terms through the *third* order in J_2, and the secular terms through the *fourth* order in J_2. In practice, so high an order may seem unrealistic. But, because the series are purely literal, they constitute a sort of *archive* document: within the accuracy of twelve significant figures, their coefficients have been determined once and for all. In the majority of cases where the second order is sufficient, the user can ignore in the series the contributions of the third order. But for very long arc predictions, it is, of course, imperative to take into account the secular terms of order three and four.

In sum, the higher the order of a literal theory, the more users it is likely to serve.

Brouwer's theory consists essentially of two successive canonical transformations. We tried to compose them into a unique transformation as Brouwer proposed it in his original paper. But the operations involved so large a number of terms in the end products as well as in the intermediate results that we concluded it would be more economic to keep the two mappings separate.

(v) The Main Problem of an artificial satellite is treated here explicitly as a problem of initial conditions. The constants of the motion are not left to be determined by successive iterations (Cain 1962) or by least squares; the inverse canonical transformations are used to develop explicitly the series that express the average elements in terms of the osculating elements.

The present communication is well restricted in its intentions. We meant more than reproducing by machine the original paper of Brouwer and its extension by Kozai. In fact we reworked its underlying formalism and saw to it that it lended itself to a smooth automatic processing by computers. At the same time, we eradicated the main sources of difficulties that so far have adversely affected the use of several theories of artificial satellites, namely, the premature truncatures in J_2, the determination of the constants of the motion, and the singularity of the zero eccentricity. Having established that the Main Problem of Brouwer's theory can be solved to the greater convenience of the users, we reckon that the completion of

Brouwer's theory now is more a matter of developmental effort than a research problem.

The basic steps of the paper are the classical ones. We expounded on the introduction of Delaunay's variables with more details than are necessary for an expert (Section 2); but, in the classroom and in seminars, we experienced that the standard references do not adequately meet with the demands of astrodynamics engineers in this matter. Sections 3 - 5 cover the elimination of short period terms, i.e., the construction of a canonical transformation to average over the mean anomaly, while Sections 6 and 7 outline the elimination of the long period terms, i.e., the averaging over the argument of perigee by constructing a second canonical transformation. We conclude the analytical study by indicating how the secular equations and the ephemeris (in position and velocity) ought to be computed so that the singularity of zero eccentricity can be radically eradicated (Sections 8 and 9).

Finally we present numerical evidence as to the reliability of the present solution. As illustrations, we took the satellites ANNA 1B and RELAY II because their orbits have recently served to compare various theories (Bonavito *et al* 1968). Positions and velocities computed from the series are compared with the results of an accurate stepwise integration of the Main Problem. The errors exhibited are unusually small, although they are accumulating over time intervals comparable with the period of rotation of the perigee. Our purpose here is only to establish that the analytical solution as we have it does not *distort* appreciably, even over long arcs, the problem it purports to solve, which is solely Brouwer's Main Problem of Satellite Theory.

1. The Main Problem

In the six-dimensional phase space product of the three-dimensional Euclidean space of positions (x,y,z) by the three-dimensional Euclidean space of velocities (X,Y,Z), consider the Hamiltonian function

$$\mathcal{H} = \frac{1}{2}\,(X^2+Y^2+Z^2) - \frac{\mu}{r}\left[1 - \frac{1}{2}\,J_2\left(\frac{R_e}{r}\right)^2(3\,\sin^2\beta - 1)\right] \qquad (1)$$

where

$$r = |x^2+y^2+z^2|^{\frac{1}{2}}$$

and β is the latitude with respect to the coordinate plane Oxy, thus unambiguously defined by the trigonometric relations

$$\cos\beta = (x^2+y^2)^{\frac{1}{2}}/r, \qquad \sin\beta = z/r.$$

The system described by the Hamiltonian (1) constitutes the Main Problem (MP) in the theory of a close satellite for an oblate planet. In that context, $\mu > 0$ is the constant of gravitation for the planet (dimension: length^3/time^2), $R_e > 0$ is the mean radius of the planet, whereas $J_2 \neq 0$ is a (dimensionless) constant of oblateness.

This is a reversible system with three degrees of freedom. Its Hamiltonian being conservative, it admits as a first integral

$$\mathcal{H} = \text{constant}. \qquad (2)$$

Moreover the same Hamiltonian being invariant with respect to the commutative group of rotations around the position axis Oz, the MP possesses as a second integral the function

$$H = xY - yX, \tag{3}$$

which is the component along Oz of the angular momentum per unit of mass with respect to O for the particle in motion relative to the coordinate system $Oxyz$.

The integral (3) is used to render ignorable one coordinate in the Hamiltonian (1). Consider the function

$$S \equiv S(x_N, y_N, h, X, Y, Z)$$

$$= -x_N(X \cos h + Y \sin h) - y_N[(-X \sin h + Y \cos h)^2 + Z^2]^{\frac{1}{2}}.$$

For the sake of brevity introduce the auxiliary function

$$I \equiv I(h, X, Y, Z)$$

unambiguously defined by the consistent trigonometric relations

$$\cos I = (-X \sin h + Y \cos h)/[(-X \sin h + Y \cos h)^2 + Z^2]^{\frac{1}{2}},$$
$$\sin I = Z/[(-X \sin h + Y \cos h)^2 + Z^2]^{\frac{1}{2}}.$$

The equations

$$x = -\partial S/\partial X = x_N \cos h - y_N \sin h \cos I,$$
$$y = -\partial S/\partial Y = x_N \sin h + y_N \cos h \cos I,$$
$$z = -\partial S/\partial Z = y_N \sin I,$$
$$X_N = -\partial S/\partial x_N = X \cos h + Y \sin h,$$
$$Y_N = -\partial S/\partial y_N = [(-X \sin h + Y \cos h)^2 + Z^2]^{\frac{1}{2}},$$
$$H = -\partial S/\partial h = xY - yX$$

define implicitly a completely canonical transformation from the state variables (x,y,z,X,Y,Z) to the state variables (x_N,y_N,h,X_N,Y_N,H). The first three equations determine a coordinate transformation from the position frame $Oxyz$ to any frame Ox_Ny_NN whose plane Ox_Ny_N contains the position vector. This transformation composes a rotation about the axis Oz with an amplitude h that carries the axis Ox onto the axis Ox_N, and a rotation about the axis Ox_N with the amplitude I that carries the plane Oxy onto the plane Ox_Ny_N. Accordingly the angle h is called the *longitude of the ascending node* Ox_N, and the angle I the *inclination* of the plane Ox_Ny_N over the plane Oxy.

The last three equations of the transformation are inverted to obtain that

$$X = X_N \cos h - Y_N \sin h \cos I,$$
$$Y = X_N \sin h + Y_N \cos h \cos I,$$
$$Z = Y_N \sin I.$$

These formulas mean that the plane Ox_Ny_N has been selected to contain permanently the velocity vector, and that (X_N,Y_N) are the components of the velocity vector in that plane. For this reason the coordinate plane Ox_Ny_N is called the *instantaneous orbital plane*; we shall denote by $\Pi(t)$.

The following identities are trivial consequences of the transformation equations:

$$r^2 = x^2 + y^2 + z^2 = x_N^2 + y_N^2,$$

$$V^2 = X^2 + Y^2 + Z^2 = X_N^2 + Y_N^2,$$

$$\underset{\sim}{x} \cdot \underset{\sim}{X} = xX + yY + zZ = x_N X_N + y_N Y_N.$$

The angular momemtum $\underset{\sim}{G} = \underset{\sim}{x} \times \underset{\sim}{X}$ per unit of mass is oriented along the coordinate axis ON normal to $\Pi(t)$, its component along that axis being

$$G = x_N Y_N - y_N X_N;$$

consequently

$$yZ - zY = G \sin I \sin h,$$

$$zX - xZ = -G \sin I \cos h,$$

$$H = xY - yX = G \cos I.$$

Also in the new variables the formulas defining the latitude become

$$\cos \beta = (x_N^2+y_N^2)^{\frac{1}{2}}/r, \qquad \sin \beta = (y_N/r)\sin I.$$

As a result of the rotation from the fixed frame $Oxyz$ to the moving frame $Ox_N y_N N$, the Hamiltonian of the MP turns out to be

$$\begin{aligned} \mathcal{H} &\equiv \mathcal{H}(x_N, y_N, -, X_N, Y_N, H) \\ &= \frac{1}{2}(X_N^2+Y_N^2) - \frac{\mu}{r}\left[1 - \frac{1}{2} J_2\left(\frac{R_e}{r}\right)^2\left(3 \frac{y_N^2}{r^2} \sin^2 I - 1\right)\right]. \end{aligned} \tag{4}$$

In the list of arguments for $\mathcal{H}$, where the coordinate h was expected, we place a dash to emphasize that it has become an ignorable coordinate.

For a satellite whose planetocentric distance r is of the order of R_e, the part of (4) which contains J_2 is of the order of J_2. Henceforth assuming that J_2 is a small number, we decompose (4) into the sum

$$\mathcal{H} = \mathcal{H}_0 + J_2\mathcal{H}_1 \tag{5_1}$$

with

$$\mathcal{H}_0 \equiv \mathcal{H}_0(x_N,y_N,-,X_N,Y_N,-) = \frac{1}{2}(X_N^2+Y_N^2) - \frac{\mu}{r}, \tag{5_2}$$

$$\mathcal{H}_1 \equiv \mathcal{H}_1(x_N,y_N,-,X_N,Y_N,H) = \frac{1}{2}\frac{\mu R_e^2}{r^3}\left(3\frac{y_N^2}{r^2}\sin^2 I-1\right). \tag{5_3}$$

In this way, the MP is interpreted as basically a problem of two bodies described by (5_2), but perturbed by forces whose potential is $J_2\mathcal{H}_1$.

Along this line of approach our interest lies in simplifying as much as possible the principal component $\mathcal{H}_0$. A first step in that direction consists in introducing the polar components in the instantaneous orbital plane $\Pi(t)$. Thus consider the function

$$S \equiv S(r,\theta,X_N,Y_N) = -r(X_N\cos\theta+Y_N\sin\theta)$$

and the completely canonical transformation from the state variables (x_N,y_N,X_N,Y_N) to the state variables (r,θ,R,Θ) implicitly defined by the equations

$$\begin{aligned}
x_N &= -\partial S/\partial X_N = r\cos\theta, \\
y_N &= -\partial S/\partial Y_N = r\sin\theta, \\
R &= -\partial S/\partial r = X_N\cos\theta + Y_N\sin\theta, \\
\Theta &= -\partial S/\partial\theta = r(-X_N\sin\theta+Y_N\cos\theta).
\end{aligned}$$

We easily check that

$$\Theta = x_N Y_N - y_N X_N,$$

$$X_N^2 + Y_N^2 = R^2 + \frac{1}{r^2}\Theta^2,$$

so that the Hamiltonian (5) transforms into

$$\mathcal{H} \equiv \mathcal{H}(r,\theta,-,R,\Theta,H) = \mathcal{H}_0 + J_2\mathcal{H}_1 \tag{6_1}$$

with

$$\mathcal{H}_0 \equiv \mathcal{H}_0(r,-,-,R,\Theta,-) = \frac{1}{2}\left(R^2 + \frac{1}{r^2}\Theta^2\right) - \frac{\mu}{r}, \tag{6_2}$$

$$\mathcal{H}_1 \equiv \mathcal{H}_1(r,\theta,-,-,\Theta,H) = \frac{1}{2}\frac{\mu R_e}{r^3}\left[\left(\frac{1}{2} - \frac{3}{2}\cos^2 I\right) - \frac{3}{2}\sin^2 I \cos 2\theta\right]. \tag{6_3}$$

We aim at simplifying further the principal part (6_2) by rendering ignorable all state variables but one action. This ultimate reduction of the problem of two bodies is accomplished by Delaunay's mapping. Consider the function

$$P \equiv P(r,L,G;\mu) = [-(\mu^2/L^2) + 2(\mu/r) - (G^2/r^2)]^{\frac{1}{2}}.$$

For fixed L and G such that

$$L > G > 0, \tag{7}$$

P taken as a function of r has two distinct roots, namely

$$r_P \equiv r_P(L,G;\mu) = [L - (L^2-G^2)^{\frac{1}{2}}]L/\mu,$$

$$r_A \equiv r_A(L,G;\mu) = [L + (L^2-G^2)^{\frac{1}{2}}]L/\mu,$$

such that $r_A > r_P > 0$; moreover, P is real under the condition that $r_P \leq r \leq r_A$, in which case it takes the form

$$P = (\mu/Lr)[(r_A-r)(r-r_L)]^{\frac{1}{2}}.$$

Under the assumption (7), consider the function

$$S \equiv S(r,\theta,L,G;\mu) = G\theta + \int_{r_P}^{r} P(\bar{r},L,G;\mu)\,d\bar{r}$$

and the completely canonical transformation from the state variables (r,θ,R,Θ) to the state variables (ℓ,g,L,G) implicitly defined by the equations

$$R = \partial S/\partial r = P(r,L,G;\mu),$$

$$\Theta = \partial S/\partial\theta = G,$$

$$\ell = \partial S/\partial L = (\mu^2/L^3)\int_{r_P}^{r} d\bar{r}/P(\bar{r},L,G;\mu),$$

$$g = \partial S/\partial G = \theta - G\int_{r_P}^{r} d\bar{r}/\bar{r}^2 P(\bar{r},L,G;\mu).$$

It can be expressed in a closed form. Indeed define the functions $a \equiv a(L,\mu)$ and $e \equiv e(L,G)$ by the conditions

$$a > 0, \qquad L^2 = \mu a,$$

$$1 > e > 0, \qquad G = L(1-e^2)^{\frac{1}{2}}.$$

In terms of a and e, the roots of $P(r) = 0$ become

$$r_P = a(1-e), \qquad r_A = a(1+e).$$

Now uniformize the quadrature for ℓ by substituting for the variable r an angle u such that

$$r = a(1-e\cos u);$$

compute that

$$dr = ae \sin u \, du$$

$$r_A - r = 2ae \cos^2(u/2),$$

$$r - r_p = 2ae \sin^2(u/2),$$

and check that the equation in ℓ becomes

$$\ell = u - e \sin u.$$

The quadrature defining g is uniformized by substituting for the variable r an angle f such that

$$1/r = (1+e \cos f)/p,$$

where $p = p(G;\mu)$ is defined by the conditions

$$p > 0, \qquad G^2 = \mu p.$$

Calculate that

$$dr = (e/p) r^2 \sin f \cdot df,$$

$$r_A - r = [2e/(1-e)] r \cos^2(f/2),$$

$$r - r_p = [2e/(1+e)] r \sin^2(f/2),$$

and so check that

$g =$ [illegible] $- f.$

From the resulting identities

$$R = (Le/r) \sin u = (\mu e/G) \sin f,$$

we easily derive that

$$R^2 + (G^2/r^2) = (2\mu/r) - (\mu^2/L^2).$$

In sum, Delaunay's mapping transforms the Hamiltonian (6) into the sum

$$\mathcal{H} \equiv \mathcal{H}(\ell,g,-,L,G,H) = \mathcal{H}_0 + J_2\mathcal{H}_1, \tag{7$_1$}$$

$$\mathcal{H}_0 \equiv \mathcal{H}_0(-,-,-,L,-,-) = -\mu^2/2L^2, \tag{7$_2$}$$

$$\mathcal{H}_1 \equiv \mathcal{H}_1(\ell,g,-,L,G,H) = \frac{1}{2}\,\frac{\mu R_e^2}{r^3}\left[\left(\frac{1}{2} - \frac{3}{2}\cos^2 I\right) - \frac{3}{2}\sin^2 I\,\sin(2f+2g)\right] \tag{7$_3$}$$

2. The Case of Small Eccentricities

We propose to expand the perturbation $\mathcal{H}_1$ as given by (7_3) in power series of the eccentricity, thus limiting the application of the theory to close satellites with small eccentricities.

The development of $1/r^3$, $\cos 2f/r^3$ and $\sin 2f/r^3$ is implemented automatically by computer (Deprit and Rom 1967). The following remarks are in order. As power series of e with coefficients being trigonometric sums in the multiples of ℓ, the functions $1/r^3$, $e^2\cos 2f/r^3$ and $e^2\sin 2f/r^3$ present the d'Alembert characteristic (Brouwer and Clemence 1961). Therefore, if we introduce the mean distance

$$F = \ell + g$$

to the ascending node, we can observe that the power series

$$\frac{\sin(2f+2g)}{r^3} = \sum_{j\geq 0} e^j \sum_k A_{j,k}\sin(k\ell+2F)$$

whose trigonometric arguments are not ℓ and g but ℓ and F presents also the d'Alembert characteristic, i.e., the summation index k satisfies both conditions:

$$|k| \leq j, \qquad k \equiv j \pmod{2}.$$

We complete the expansion of $\mathcal{H}_1$ by observing that

$$\cos I = H/G = (H/L)(1-e^2)^{-\frac{1}{2}},$$

thus making use of the binomial series to develop $\cos I$ in powers of e. Eventually trivial manipulations of power series in e will bring $\mathcal{H}_1$ in the form

$$\mathcal{H}_1 = \mu^4 R_e^2 L^{-6} \sum_{j \geq 0} e^j \left[\mathcal{H}_{1,j,0} + \mathcal{H}_{1,j,2} (H/L)^2\right]$$

where, for any $j \geq 0$, the coefficients $\mathcal{H}_{1,j,0}$ and $\mathcal{H}_{1,j,2}$ are finite sums of cosine functions with arguments of the type $p\ell + 2kF$, the multiples k and p satisfying the conditions

$$k = 0 \text{ or } 1, \qquad |p| \leq j, \qquad p \equiv j \pmod{2}.$$

To illustrate the development, we have edited in Table I the results up to degree 5 in e obtained by computer. We have omitted the factor

$$\mu^4 R_e^2 L^{-6} = n^2 a^2 (R_e/a)^2 = n^2 R_e^2.$$

The various trigonometric arguments are entered in Column 1; the cosines are multiplied already by the smallest power of e compatible with the d'Alembert characteristic. Then the second column lists the principal parts of the coefficients, the next column the parts of degree e^2, and so on.

Table I. Expansion of the Perturbation Potential $\mathcal{H}_1$

		e^2	e^4
Ind. term	$\frac{1}{4} - \frac{3}{4}\eta^2$	$\frac{3}{8} - \frac{15}{8}\eta^2$	$\frac{5}{32} - \frac{105}{32}\eta^2$
cos 2F	$-\frac{3}{4} + \frac{3}{4}\eta^2$	$\frac{15}{8} - \frac{9}{8}\eta^2$	$-\frac{39}{64} - \frac{165}{32}\eta^2$
$e \cos \ell$	$\frac{3}{4} - \frac{9}{4}\eta^2$	$\frac{19}{32} - \frac{153}{32}\eta^2$	
$e \cos(\ell+2F)$	$-\frac{21}{8} + \frac{21}{8}\eta^2$	$\frac{369}{64} - \frac{201}{64}\eta^2$	
$e \cos(\ell-2F)$	$\frac{3}{8} - \frac{3}{8}\eta^2$	$-\frac{3}{64} - \frac{21}{64}\eta^2$	
$e^2 \cos 2\ell$	$\frac{9}{8} - \frac{27}{8}\eta^2$	$\frac{7}{8} - 6\eta^2$	
$e^2 \cos(2\ell+2F)$	$-\frac{51}{8} + \frac{51}{8}\eta^2$	$\frac{115}{8} - 8\eta^2$	
$e^2 \cos(2\ell-2F)$	M I S S I N G		
$e^3 \cos 3\ell$	$\frac{53}{32} + \frac{159}{32}\eta^2$		
$e^3 \cos(3\ell+2F)$	$-\frac{845}{64} + \frac{845}{64}\eta^2$		
$e^3 \cos(3\ell-2F)$	$-\frac{1}{64} + \frac{1}{64}\eta^2$		
$e^4 \cos 4\ell$	$\frac{77}{32} - \frac{231}{32}\eta^2$		
$e^4 \cos(4\ell+2F)$	$-\frac{1599}{64} + \frac{1599}{64}\eta^2$		
$e^4 \cos(4\ell-2F)$	$-\frac{1}{32} + \frac{1}{32}\eta^2$		

As we shall see later, the theory of satellites depends vitally on the fact that the expansion of $\mathcal{H}_1$ does not contain the argument $2g = 2F - 2\ell$. This is proved to be the case by computing that

$$\langle \mathcal{H}_1 \rangle_\ell = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}_1 d\ell = \frac{1}{2}\,\mu^4 R_e^2 L^{-3} G^{-3}\left(1-3\,\frac{H^2}{G^2}\right).$$

The d'Alembert character we just emphasized will also prove quite useful. Indeed the perturbation algorithm we are going to use will consist exclusively of Poisson brackets, formal quadratures and averagings involving series that will all derive from $\mathcal{H}_1$. But the algebra of series having the d'Alembert characteristic is closed for these operations. Thus by checking that the initial input $\mathcal{H}_1$ has this character, we make certain that the formalism will produce only functions having the same character. In particular, we do not have to make provision for negative powers of the eccentricity.

3. Elimination of the Short Period Terms

We propose to build a completely canonical transformation from the osculating Delaunay's state coordinates (ℓ,g,h,L,G,H) to new coordinates (ℓ',g',h',L',G',H') such that, in the transformed Hamiltonian, the variable ℓ' becomes ignorable. We want the transformation in power series of J_2. To this effect we shall use the perturbation technique based on Lie transforms (Deprit 1969).

As we propose to truncate the development after J_2^3, we need only to fill in the triangle

$$
\begin{array}{ccccccccc}
 & & & & \mathcal{H}_0^0 & & & & \\
 & & & \mathcal{H}_1^0 & & \mathcal{H}_0^1 & & & \\
 & & \mathcal{H}_2^0 & & \mathcal{H}_1^1 & & \mathcal{H}_0^2 & & \\
 & \mathcal{H}_3^0 & & \mathcal{H}_2^1 & & \mathcal{H}_1^2 & & \mathcal{H}_0^3 & \\
\mathcal{H}_4^0 & & \mathcal{H}_3^1 & & \mathcal{H}_2^2 & & \mathcal{H}_1^3 & & \mathcal{H}_0^4
\end{array}
$$

The last line is to be computed in anticipation of the long period elimination that we shall execute later on.

We enter the triangle by putting

$$
\begin{aligned}
\mathcal{H}_0^0 &= \mathcal{H}_0(-,-,-,L',-,-), \\
\mathcal{H}_1^0 &= \mathcal{H}_1(\ell',g',-,L',G',H'), \\
\mathcal{H}_n^0 &= 0 \qquad\qquad \text{for} \quad n \geq 2.
\end{aligned}
$$

Let us indicate by

$$
(\phi;\psi) = \left(\frac{\partial\phi}{\partial\ell'} \cdot \frac{\partial\psi}{\partial L'} - \frac{\partial\phi}{\partial L'} \cdot \frac{\partial\psi}{\partial\ell'}\right) + \left(\frac{\partial\phi}{\partial g'} \cdot \frac{\partial\psi}{\partial G'} - \frac{\partial\phi}{\partial G'} \cdot \frac{\partial\psi}{\partial g'}\right) +
$$

$$
+ \left(\frac{\partial\phi}{\partial h'} \cdot \frac{\partial\psi}{\partial H'} - \frac{\partial\phi}{\partial H'} \cdot \frac{\partial\psi}{\partial h'}\right)
$$

the Poisson bracket of the functions ϕ and ψ in the phase space (ℓ',g',h',L',G',H'). The calculations of the Poisson brackets require in the present case some attention. We have to deal with functions whose list of arguments are formally

$$
L',e',H',\ell',g'.
$$

Thus, on one hand, because the variable h' is ignorable in both ϕ and ψ, the calculation of $(\phi;\psi)$ will reduce to the first two Jacobians. On the other hand, L' appears in two places, among the explicit variables, but also implicitly in e', whereas G' appears not among the explicit arguments but only implicitly in e'. Therefore by differentiation in chain we have that

$$\frac{\partial\phi}{\partial G'} = \frac{\partial\phi}{\partial e'} \cdot \frac{\partial e'}{\partial G'} = - \frac{(1-e'^2)^{\frac{1}{2}}}{L'e'} , \tag{8_1}$$

$$\frac{\partial\phi}{\partial L'} = \left(\frac{\partial\phi}{\partial L'}\right)_{e'} + \frac{\partial\phi}{\partial e'} \cdot \frac{\partial e'}{\partial L'} = \left(\frac{\partial\phi}{\partial L'}\right)_{e'} + \frac{1 - e'^2}{L'e'} . \tag{8_2}$$

The square root appearing in $\partial\phi/\partial G'$ will of course be replaced by its binomial expansion in powers of e'. Assume that the functions ϕ and ψ have the d'Alembert characteristic we already mentioned; then the above formulas indicate that their partial derivatives with respect to L' and G' lose this characteristic, and worse yet, contain terms in e'^{-1}. Nevertheless, the Poisson bracket $(\phi;\psi)$ retains the d'Alembert characteristic (Brown and Shook 1933). For in the course of calculating the Jacobians

$$\frac{\partial\phi}{\partial \ell'} \frac{\partial\psi}{\partial L'} - \frac{\partial\phi}{\partial L'} \cdot \frac{\partial\psi}{\partial \ell'} \quad \text{and} \quad \frac{\partial\phi}{\partial g'} \cdot \frac{\partial\psi}{\partial G'} - \frac{\partial\phi}{\partial G'} \cdot \frac{\partial\psi}{\partial g'}$$

the terms in e'^{-1} cancel one another. If the programming techniques for manipulating literal expressions by computers enable one to process polynomial variables with negative exponents--such is the case for MAO (Rom 1969)--it is relatively easy to write subroutines that will implement the partial differentiations $\partial/\partial G'$, $\partial/\partial L'$ as written in (8_1) and (8_2). Then the natural cancellation in the Poisson bracket $(\phi;\psi)$ of the terms in e'^{-1} may serve as a check on the validity of the coding while in the initial phase of debugging the program, and as a test of the absolute accuracy on the coefficients obtained as soon as the program becomes operational.

Notice that for any function $\phi(\ell',g',h',L',G',H')$,

$$(\mathcal{H}_0^0;\phi) = -\mu^2 L'^{-3}(\partial\phi/\partial\ell').$$

After these preliminaries, we outline the operations that accomplished the elimination of the short period terms.

Order 1. The basic identity being

$$\mathcal{H}_1^0 + (\mathcal{H}_0^0;W_1) = \mathcal{H}_0^1,$$

we selected

$$\mathcal{H}_0^1 = \langle\mathcal{H}_1^0\rangle_{\ell'}.$$

Then putting

$$\mathcal{P}_1 = \mathcal{H}_1^0 - \mathcal{H}_0^1,$$

we obtained the generator W_1 by the quadrature

$$W_1 = \mu^{-2}L'^3\int^{\ell'} \mathcal{P}_1\,d\ell'.$$

Averaging and quadrature are two operations simple to program in MAO's language. The average of a function ϕ periodic in ℓ' is obtained by transferring from the Poisson series ϕ into the Poisson series $\langle\phi\rangle$ the terms whose trigonometric argument does not contain ℓ'. Then the terms left in ϕ constitute the properly periodic part of ϕ; let us call it $\mathcal{P}$. As $\mathcal{P}$ contains no constant term, the quadrature

$$W = \int^{\ell'} \mathcal{P} d\ell'$$

is a Poisson series having the same type as $\mathcal{P}$. A term like $\cos(p\ell'+\alpha)$ of $\mathcal{P}$ is transferred into W as the term $(1/p)\sin(p\ell'+\alpha)$; similarly a term like $\sin(p\ell'+\alpha)$ of $\mathcal{P}$ is transferred into W as the term $(-1/p)\cos(p\ell'+\alpha)$. Table II lists the development of W up to e^5.

The final result is a first order generator of the form

$$W_1 \equiv W_1(\ell',g',-,L',G',H';R_e,\mu)$$

$$= \mu^2 R_e^2 L'^{-3} \sum_{j\geq 0} e'^j [W_{1,j,0} + W_{1,j,2}(H'/L')^2]$$

where, for any $j \geq 0$, the coefficients $W_{1,j,0}$ and $W_{1,j,2}$ are finite sums of sine functions in the arguments $p\ell' + 2kF'$ with $k = 0$ or 1 and the d'Alembert characteristics $|p| \leq j$, $p \equiv j \pmod{2}$. As for the first order component $\mathcal{H}_0^1$ in the averaged Hamiltonian, we obtain

$$\mathcal{H}_0^1 \equiv \mathcal{H}_0^1(-,-,-,L',G',H';R_e,\mu)$$

$$= \mu^4 R_e^2 L'^{-6} \sum_{j\geq 0} e'^{2j} [\mathcal{H}^1_{0,j,0} + \mathcal{H}^1_{0,j,2}(H'/L')^2],$$

where, for any $j \geq 0$, the coefficients $\mathcal{H}^1_{0,j,0}$ and $\mathcal{H}^1_{0,j,2}$ are purely numerical.

Incidentally, until we reached the end products of the transformation, we decided to keep explicitly among the polynomial symbols of our Poisson series the quantities μ and R_e, thus not availing ourselves of a natural system of units like the Vanguard units to render dimensionless Delaunay's action momenta. Indeed it proved useful to constantly monitor

Table II. The first order generator W_1

(The expansion must be multiplied by $\mu^2 R_e^2 L'^{-3}$)

		e'^2	e'^4
$\sin 2F'$	$-\frac{3}{8}+\frac{3}{8}\eta'^2$	$\frac{15}{16}-\frac{9}{16}\eta'^2$	$-\frac{39}{128}-\frac{33}{128}\eta'^2$
$e'\sin \ell'$	$\frac{3}{4}-\frac{9}{4}\eta'^2$	$\frac{27}{32}-\frac{153}{32}\eta'^2$	$\frac{261}{256}-\frac{2007}{256}\eta'^2$
$e'\sin(\ell'+2F')$	$-\frac{7}{8}+\frac{7}{8}\eta'^2$	$\frac{123}{64}-\frac{67}{64}\eta'^2$	$-\frac{489}{512}-\frac{47}{512}\eta'^2$
$e'\sin(\ell'-2F')$	$-\frac{3}{8}+\frac{3}{8}\eta'^2$	$\frac{3}{64}+\frac{21}{64}\eta'^2$	$-\frac{1}{1024}+\frac{173}{512}\eta'^2$
$e'^2\sin 2\ell'$	$\frac{9}{16}-\frac{27}{16}\eta'^2$	$\frac{7}{16}-3\eta'^2$	
$e'^2\sin(2\ell'+2F')$	$-\frac{51}{32}+\frac{51}{32}\eta'^2$	$\frac{115}{32}-2\eta'^2$	
$e'^2\sin(2\ell'-2F')$	MISSING		
$e'^3\sin 3\ell'$	$\frac{53}{96}-\frac{53}{32}\eta'^2$	$\frac{131}{512}-\frac{1241}{512}\eta'^2$	
$e'^3\sin(3\ell'+2F')$	$-\frac{169}{64}+\frac{169}{64}\eta'^2$	$\frac{6505}{1024}-\frac{3801}{1024}\eta'^2$	
$e'^3\sin(3\ell'-2F')$	$\frac{1}{64}-\frac{1}{64}\eta'^2$	$-\frac{11}{1024}+\frac{27}{1024}\eta'^2$	
$e'^4\sin 4\ell'$	$\frac{77}{128}-\frac{231}{128}\eta'^2$		
$e'^4\sin(4\ell'+2F')$	$-\frac{533}{128}+\frac{533}{128}\eta'^2$		
$e'^4\sin(4\ell'-2F')$	$-\frac{1}{64}+\frac{1}{64}\eta'^2$		
$e'^5\sin 5\ell'$	$\frac{1773}{2560}-\frac{5319}{2560}\eta'^2$		
$e'^5\sin(5\ell'+2F')$	$-\frac{32621}{5120}+\frac{32621}{5120}\eta'^2$		
$e'^5\sin(5\ell'-2F')$	$-\frac{81}{5120}+\frac{81}{5120}\eta'^2$		

the validity of our codings as well as the operational condition of the equipment by inspecting the physical dimensions of the state variables as they were being generated. For instance, in the present problem, all elements $\mathcal{H}_k^j$ of the basic Lie triangle have the dimensions of an energy per unit of mass (i.e., length^2/time^2) whereas the generators W_j have those of an action per unit of mass (i.e., length^2/time).

Order 2. Until the moment comes to compute it, the second order generator W_2 is assumed to be the null Poisson series. Under this assumption we calculate in the Lie triangle the provisional elements

$$\tilde{\mathcal{H}}_1^1 = \mathcal{H}_2^0 + (\mathcal{H}_1^0;W_1) + (\mathcal{H}_0^0;W_2) = (\mathcal{H}_1^0;W_1),$$

$$\tilde{\mathcal{H}}_0^2 = \tilde{\mathcal{H}}_1^1 + (\mathcal{H}_0^1;W_1).$$

Thus we arrive at the second order differential identity

$$\tilde{\mathcal{H}}_0^2 + (\mathcal{H}_0^0;W_2) = \mathcal{H}_0^2.$$

We select for $\mathcal{H}_0^2$ the terms of $\tilde{\mathcal{H}}_0^2$ independent from ℓ', we put

$$\mathcal{P}_2 \equiv \mathcal{P}_2(\ell',g',-,L',G',H') = \tilde{\mathcal{H}}_0^2 - \mathcal{H}_0^2$$

and we obtain the second order generator through the quadrature

$$W_2 = \mu^{-2}L'^3\int^{\ell'}\mathcal{P}_2\,d\ell'.$$

It turns out to be a series of the form

$$W_2 \equiv W_2(\ell',g',-,L',G',H';R_e,\mu)$$

$$= \mu^4R_e^4L'^{-7}\sum_{j\geq 0} e'^j[W_{2,j,0}+W_{2,j,2}(H'/L')^2+W_{2,j,4}(H'/L')^4]$$

Table III. The second order generator W_2

(The expansion must be multiplied by $\mu^4 R_e^4 L'^{-7}$)

		e'^2
$\sin 2F'$	$\frac{3}{16} + \frac{3}{8}\eta'^2 - \frac{9}{16}\eta'^4$	$-\frac{459}{128} + \frac{705}{32}\eta'^2 - \frac{2457}{128}\eta'^4$
$\sin 4F'$	$\frac{3}{64} - \frac{3}{32}\eta'^2 - \frac{3}{64}\eta'^4$	$-\frac{3}{8} + \frac{21}{32}\eta'^2 - \frac{9}{32}\eta'^4$
$e'\sin \ell'$	$\frac{39}{128} + \frac{9}{64}\eta'^2 - \frac{777}{128}\eta'^4$	
$e'\sin(\ell'+2F')$	$\frac{85}{64} - \frac{33}{8}\eta'^2 + \frac{179}{64}\eta'^4$	
$e'\sin(\ell'-2F')$	$\frac{45}{32} - \frac{135}{16}\eta'^2 + \frac{225}{32}\eta'^4$	
$e'\sin(\ell'+4F')$	$\frac{15}{64} - \frac{15}{132}\eta'^2 + \frac{15}{64}\eta'^4$	
$e'\sin(\ell'-4F')$	$\frac{9}{128} - \frac{9}{64}\eta'^2 + \frac{9}{128}\eta'^4$	
$e'^2\sin 2\ell'$	$\frac{105}{256} - \frac{63}{128}\eta'^2 - \frac{1227}{256}\eta'^4$	
$e'^2\sin(2\ell'+2F')$	$\frac{459}{128} - \frac{891}{64}\eta'^2 + \frac{1323}{128}\eta'^4$	
$e'^2\sin(2\ell'-2F')$	MISSING	
$e'^2\sin(2\ell'+4F')$	$\frac{189}{256} - \frac{189}{128}\eta'^2$	
$e'^2\sin(2\ell'-4F')$	$-\frac{3}{128} + \frac{3}{64}\eta'^2 - \frac{3}{128}\eta'^4$	

where, for any $j \geq 0$, the coefficients $W_{2,j,0}$, $W_{2,j,2}$ and $W_{2,j,4}$ are finite sums of sine functions in the arguments $p\ell' + 2kF'$ with $k = 0$, 1 or 2 and the d'Alembert characteristic $|p| \leq j$, $p \equiv j \pmod{2}$. The second order part $\mathcal{H}_0^2$ in the averaged Hamiltonian is of the type

$$\mathcal{H}_0^2 = \mathcal{H}_0^2(-,g',-,L',G',H';R_e,\mu)$$

$$= \mu^6 R_e L'^{-10} \sum_{j\geq 0} e'^j [\mathcal{H}^2_{0,j,0} + \mathcal{H}^2_{0,j,2}(H'/L')^2 + \mathcal{H}^2_{0,j,4}(H'/L')^4]$$

where, for any $j \geq 0$, the coefficients $\mathcal{H}^2_{0,j,0}$, $\mathcal{H}^2_{0,j,2}$ and $\mathcal{H}^2_{0,j,4}$ are sums of cosine functions in the argument $2kg'$ with $k = 0$ or 1.

<u>Order 3</u>. The element $\tilde{\mathcal{H}}_1^1$ in the triangle can now be completed; thus we compute

$$\mathcal{H}_1^1 = \tilde{\mathcal{H}}_1^1 - \mathcal{P}_2.$$

We begin the calculations at order 3 by putting $W_3 = 0$, and we compute under this assumption the elements in the fourth row of the triangle:

$$\tilde{\mathcal{H}}_2^1 = \mathcal{H}_3^0 + (\mathcal{H}_2^0;W_1) + 2(\mathcal{H}_1^0;W_2) + (\mathcal{H}_0^0;W_3) = 2(\mathcal{H}_1^0;W_2),$$
$$\tilde{\mathcal{H}}_2^2 = \tilde{\mathcal{H}}_2^1 + (\mathcal{H}_1^1;W_1) + (\mathcal{H}_0^1;W_2)$$
$$\tilde{\mathcal{H}}_0^3 = \tilde{\mathcal{H}}_1^2 + (\mathcal{H}_0^2;W_1)$$

Then we treat the differential identity

$$\mathcal{H}_0^3 + (\mathcal{H}_0^0,W_3) = \mathcal{H}_0^3$$

as we have already done twice before. $\mathcal{H}_0^3$ takes from $\tilde{\mathcal{H}}_0^3$ the terms independent from ℓ'; then, if

$$\mathcal{P}_3 = \tilde{\mathcal{H}}_0^3 - \mathcal{H}_0^3,$$

Table IV. The third order generator W_3

(The expansion has to be multiplied by $\mu^6 R^6 L'^{-11}$

sin 2F'	$-\frac{471}{1024} + \frac{8757}{1024}\eta'^2 - \frac{30741}{1024}\eta'^4 + \frac{22455}{1024}\eta'^6$
sin 4F'	$\frac{147}{128} - \frac{1905}{256}\eta'^2 + \frac{183}{16}\eta'^4 - \frac{1317}{256}\eta'^6$
sin 6F'	$-\frac{27}{1024} + \frac{81}{1024}\eta'^2 - \frac{81}{1024}\eta'^4 + \frac{27}{1024}\eta'^6$
e'sin ℓ'	$\frac{3}{52} - \frac{8691}{128}\eta'^2 + \frac{12051}{64}\eta'^4 - \frac{21759}{128}\eta'^6$
e'sin(ℓ'+2F')	$\frac{65}{512} + \frac{10369}{512}\eta'^2 - \frac{57677}{512}\eta'^4 + \frac{47243}{512}\eta'^6$
e'sin(ℓ'-2F')	$\frac{1089}{256} - \frac{8973}{256}\eta'^2 + \frac{17163}{256}\eta'^4 - \frac{9279}{256}\eta'^6$
e'sin(ℓ'+4F')	$\frac{17367}{2560} - \frac{122319}{2560} + \frac{192357}{2560}\eta'^4 - \frac{17517}{512}\eta'^6$
e'sin(ℓ'-4F')	$\frac{1601}{512} - \frac{15133}{512}\eta'^2 - \frac{27463}{512}\eta'^4 - \frac{11931}{512}\eta'^6$
e'sin(ℓ'+6F')	$-\frac{27}{128} + \frac{81}{512}\eta'^2 - \frac{81}{512}\eta'^4 - \frac{27}{512}\eta'^6$
e'sin(ℓ'-6F')	$-\frac{27}{512} + \frac{81}{512}\eta'^2 - \frac{81}{512}\eta'^4 + \frac{27}{512}\eta'^6$

the third order generator W_3 is derived from the quadrature

$$W_3 = \mu^{-2}L'^3\int^{\ell'} \mathcal{P}_3 d\ell'.$$

The result is a series

$$W_3 \equiv W_3(\ell',g',-,L',G',H';\mu,R_e)$$

$$= \mu^6 R_e^6 L'^{-11} \sum_{j\geq 0} e'^j [W_{3,j,0}+W_{3,j,2}(H'/L')^2+W_{3,j,4}(H'/L')^4$$

$$+W_{3,j,6}(H'/L')^6],$$

the coefficients $W_{3,j,0}$, $W_{3,j,2}$, $W_{3,j,4}$, $W_{3,j,6}$ being finite sums of sine functions in the arguments $p\ell' + 2kF'$ such that $0 \leq k \leq 6$, $|p| \leq j$ and $p \equiv j$ (mod. 2).

Order 4. As we shall see later on, in order to find the third order contributions to the long period and secular terms, we need to know the fourth order component in the Hamiltonian averaged over the mean anomaly ℓ. Consequently we have computed partially the elements in the fifth row of the triangle, namely

$$\tilde{\mathcal{H}}_3^1 = \mathcal{H}_4^0 + (\mathcal{H}_3^0;W_1) + 3(\mathcal{H}_2^0;W_2) + 3(\mathcal{H}_1^0;W_3) + (\mathcal{H}_0^0;W_4)$$

$$= 3(\mathcal{H}_1^0;W_3),$$

$$\tilde{\mathcal{H}}_2^2 = \tilde{\mathcal{H}}_3^1 + (\mathcal{H}_2^1;W_1) + 2(\mathcal{H}_1^1;W_2) + (\mathcal{H}_0^1;W_3),$$

$$\tilde{\mathcal{H}}_1^3 = \tilde{\mathcal{H}}_2^2 + (\mathcal{H}_1^2;W_1) + (\mathcal{H}_0^2;W_2);$$

then, arriving at the differential identity

Table V. Recapitulation of the Hamiltonian averaged over the short period angle and of the generators of the averaging transformation.

	$\mathcal{H}_1^0$	W_1	$\mathcal{H}_0^1$	W_2	$\mathcal{H}_0^2$	W_3	$\mathcal{H}_0^3$	W_4	$\mathcal{H}_0^4$
e'^0	4	2	2	6	3	12	4	20	5
e'^1	6	6	0	15	0	28	0	45	0
e'^2	8	6	2	18	6	36	8	60	10
e'^3	12	12	0	29	0	56	0	90	0
e'^4	14	12	2	30	6	60	12	100	15
e'^5	18	18	0	45	0	84	0	135	0
e'^6	20	18	2	45	6	84	12	140	20
e'^7	24	24	0	60	0	112	0	180	0
e'^8	26	24	2	60	6	112	12	180	20
e'^9	30	30	0	75	0	140	0	225	0
e'^{10}	32	30	2	75	6	140	12	225	20
e'^{11}	36	36	0	90	0	168	0		
e'^{12}	38	36	2	90	6	168	12		
e'^{13}	42	42	0	105	0				
e'^{14}	44	42	2	105	6				
e'^{15}	48	48	0						
e'^{16}	50	48	2						
Total	452	444	18	848	45	1200	72	1400	90

$$\tilde{\mathcal{H}}_0^4 + (\mathcal{H}_0^0;W_4) = \mathcal{H}_0^4,$$

we determined $\mathcal{H}_0^4$ by passing to it the terms of $\tilde{\mathcal{H}}_0^4$ that do not depend on ℓ'.

The constructions we just outlined have been implemented automatically, using MAO as algebraic processor. The development of $\mathcal{H}_1$ in power of e had been truncated after degree 16, for no other reason than to set a limit. At each order of the elimination, two degrees in e are lost through the differentiation with respect to G' involved in the Poisson brackets. This way we know $\mathcal{H}_0^1$ up to e^{16}, but $\mathcal{H}_0^2$ up to e^{14}, $\mathcal{H}_0^3$ up to e^{12} and $\mathcal{H}_0^4$ up to e^{10} only. By listing the number of terms in the various components calculated so far, Table III purports to suggest the size of the programming chores and how fast the elimination gains in complexity as the order increases.

4. The Elements of the Short Period Elimination

The generators W_1, W_2, W_3 determine a completely canonical transformation from the osculating elements $u = (\ell,g,h,L,G,H)$ to a new set of variables u' (ℓ',g',h',L',G',H'), the equations being of the form

$$u = u' + J_2 u_0^1(u') + \frac{1}{2} J_2^2 u_0^2(u') + \frac{1}{6} J_2^3 u_0^3(u') + \dots$$

Since the coordinate h' is ignorable in the generators,

$$H = H'.$$

On the other hand the partial derivatives of the generators with respect to L' and G' do not have the d'Alembert characteristic in reference to the pair $(e'\ell')$; as a matter of fact, they will contain negative powers of e', which is a reflection of the fact that, for circular orbits, the direction of the perigee is undetermined. But such divisors do not appear in the developments that express the mean distance to the ascending node

$$F = \ell + g$$

and the state functions

$$C = e \cos g, \qquad S = e \sin g$$

in terms of the new variables $(\ell', g', h', L', G', H')$. Therefore we propose to calculate the ephemeris of the satellite by means of the following elements: F, h, C, S, L and H. For all of them except the last one, we have obtained their expansions in powers of J_2, the coefficients coming in power series of e'.

We review here the formal characteristics of these series; the statistics of Table VI make it plain that we cannot think of reproducing them in print. For the sake of brevity, we have put

$$\eta' = H'/L', \qquad \alpha' = a'/R_\epsilon .$$

Also the upper limits mentioned in the signs of summation are merely anecdotic; these are the degrees of the eccentricity e' after which we have truncated the expansions.

a) <u>The mean distance F to the ascending node</u>

$$F = F' + J_2\alpha'^{-2}F_1' + \frac{1}{2}J_2^2\alpha'^{-4}F_2' + \frac{1}{6}J_2^3\alpha'^{-6}F_3',$$

$$F_1' = \sum_{0\le j\le 14} e'^j[F'_{1,j,0}+F'_{1,j,2}\eta'^2],$$

$$F_2' = \sum_{0\le j\le 12} e'^j[F'_{2,j,0}+F'_{2,j,2}\eta'^2+F'_{2,j,4}\eta'^4],$$

$$F_3' = \sum_{0\le j\le 10} e'^j[F'_{3,j,0}+F'_{3,j,2}\eta'^2+F'_{3,j,4}\eta'^4+F'_{3,j,6}\eta'^6].$$

The coefficients $F'_{1,j,0}$, $F'_{1,j,2}$, ..., $F'_{3,j,6}$ are finite sums of sine functions in the arguments $p\ell' + 2qF'$ such that in $F'_{r,j,k}$, we have the conditions $0 \le q \le r$, $|p| \le j$ and $p \equiv j \pmod{2}$.

b) <u>The longitude h of the ascending node</u>

$$h = h' + \eta'(J_2\alpha'^{-2}h_1' + \frac{1}{2}J_2^2\alpha'^{-4}h_2' + \frac{1}{6}J_2^3\alpha'^{-6}h_3'),$$

$$h_1' = \sum_{0\le j\le 16} e'^j h'_{1,j,0},$$

$$h_2' = \sum_{0\le j\le 14} e'^j(h'_{2,j,0}+h'_{2,j,2}\eta'^2),$$

$$h_3' = \sum_{0\le j\le 12} e'^j(h'_{3,j,0}+h'_{3,j,2}\eta'^2+h'_{3,j,4}\eta'^4).$$

The coefficients $h'_{1,j,0}$, $h'_{2,j,0}$, ..., $h'_{3,j,4}$ are finite sums of sine functions in the arguments $p\ell' + 2qF'$ such that in $h'_{r,j,k}$, we have $0 \le q \le r$, $|p| \le j$ and $p \equiv j \pmod{2}$.

c) <u>The scalar $C = e \cos g$</u>

$$C = C' + J_2\alpha'^{-2}C_1' + \frac{1}{2}J_2^2\alpha'^{-4}C_2' + \frac{1}{6}J_2^3\alpha'^{-6}C_3',$$

$$C_1' = \sum_{0\le j\le 14} e'^j(C'_{1,j,0}+C'_{1,j,2}\eta'^2),$$

$$C'_2 = \sum_{0\le j\le 12} e'^j(C'_{2,j,0}+C'_{2,j,2}\eta'^2+C'_{2,j,4}\eta'^4),$$

$$C'_3 = \sum_{0\le j\le 10} e'^j(C'_{3,j,0}+C'_{3,j,2}\eta'^2+C'_{3,j,4}\eta'^4+C'_{3,j,6}\eta'^6).$$

The coefficients $C'_{1,j,0}$, $C'_{1,j,2}$, ..., $C'_{3,j,6}$ are finite sums of cosine functions in the arguments $p\ell' + (2q+1)F'$ such that, in $C'_{r,j,k}$ we have $0 \le q \le r$, $|p| \le j$, $p \equiv j$ (mod. 2).

d) <u>The scalar $S = e \sin g$</u>

$$S = S' + J_2\alpha'^{-2}S'_1 + \frac{1}{2} J_2^2\alpha'^{-4}S'_2 + \frac{1}{6} J_2^3\alpha'^{-6}S'_6,$$

$$S'_1 = \sum_{0\le j\le 14} e'^j(S'_{1,j,0}+S'_{1,j,2}\eta'^2),$$

$$S'_2 = \sum_{0\le j\le 12} e'^j(S'_{2,j,0}+S'_{2,j,2}\eta'^2+S'_{2,j,4}\eta'^4),$$

$$S'_3 = \sum_{0\le j\le 10} e'^j(S'_{3,j,0}+S'_{3,j,2}\eta'^2+S'_{3,j,4}\eta'^4+S'_{3,j,6}\eta'^6).$$

The coefficients $S'_{1,j,0}$, $S'_{1,j,2}$, ..., $S'_{3,j,6}$ are finite sums of sine functions in the arguments $p\ell' + (2q+1)F'$ such that, in $S'_{r,j,k}$ we have $0 \le q \le r$, $|p| \le j$, $p \equiv j$ (mod. 2).

e) <u>The action L</u>

$$L = L'(1+J_2\alpha'^{-2}L'_1 + \frac{1}{2} J_2^2\alpha'^{-4}L'_2 + \frac{1}{6} J_2^3\alpha'^{-6}L'_3),$$

$$L'_1 = \sum_{0\le j\le 16} e'^j(L'_{1,j,0}+L'_{1,j,2}\eta'^2),$$

$$L'_2 = \sum_{0\le j\le 14} e'^j(L'_{2,j,0}+L'_{2,j,2}\eta'^2+L'_{2,j,4}\eta'^4),$$

$$L'_3 = \sum_{0\le j\le 12} e'^j(L'_{3,j,0}+L'_{3,j,2}\eta'^2+L'_{3,j,4}\eta'^4+L'_{3,j,6}\eta'^6).$$

The coefficients $L'_{1,j,0}$, $L'_{1,j,2}$, ..., $L'_{3,j,6}$ are finite sums of cosine functions in the arguments $p\ell' + 2qF'$ such that, in $L'_{r,j,k}$ we have $0 \leq q \leq r$, $|p| \leq j$ and $p \equiv j \pmod{2}$.

Given the values of F', h', C', S', L' and H' at an instant t, the preceding series enable us to evaluate the corresponding functions F, h, C, S, L and H at that instant of time. Indeed the decomposition

$$p\ell' + 2qF' = (2q+p)F' - pg'$$

implies that in all the coefficients, the elements e' and g' enter only through polynomials in C' and S'. For instance, a term like $e'^5\cos(3\ell'+4F')$ can be evaluated as follows:

$$e'^5\cos(3\ell'+4F') = e'^2[(e'^3\cos 3g')\cos 7F'+(e'^3\sin 3g')\sin 7F']$$

wherein

$$e'^2 = C'^2 + S'^2,$$

$$e'^3\cos 3g' = 4C'^3 - 3C',$$

$$e'^3\sin 3g' = S'(4C'^2-1).$$

Thus nowhere is it needed to evaluate separately the eccentricity e' and the argument of perigee g'. In sum, as far as the short period perturbations are concerned, the above series overcome entirely the indetermination in the direction of the perigee which is inherent to circular and almost circular orbits.

Table VI. Recapitulation of the series necessary and sufficient to evaluate the short period perturbations

$J_2 a'^{-2}$	e'	ΔF'	Δh'	ΔC'	ΔS'	ΔL'
1	0	2	1	4	4	2
	1	6	3	6	6	6
	2	6	3	12	12	6
	3	12	6	12	12	12
	4	12	6	20	20	12
	5	18	9	20	20	18
	6	18	9	28	28	18
	7	24	12	28	28	24
	8	24	12	36	36	24
	9	30	15	36	36	30
	10	30	15	44	44	30
	11	36	18	44	44	36
	12	36	18	52	52	36
	13	42	21	52	52	42
	14	42	21	60	60	42
	15		24			48
	16		24			48
		338	217	454	454	434
2	0	6	4	9	9	9
	1	15	10	18	18	15
	2	21	13	27	27	24
	3	30	19	36	36	30
	4	36	21	45	45	39
	5	45	30	51	51	45
	6	51	31	63	63	54
	7	60	40	69	69	60
	8	66	41	81	81	69
	9	75	50	87	87	75
	10	81	51	99	99	84
	11	90	60	105	105	90
	12	96	61	117	117	99
	13		70			105
	14		71			114
		672	572	807	807	912
3	0	12	9	16	16	16
	1	28	21	32	32	28
	2	40	30	48	48	44
	3	56	42	64	64	56
	4	68	51	80	80	72
	5	84	63	96	96	84
	6	96	72	112	112	100
	7	112	83	128	128	112
	8	124	92	142	142	128
	9	140	104	160	160	140
	10	152	114	175	175	156
	11		125			168
	12		135			184
		912	941	1053	1053	1288
	Total	1922	1730	2314	2314	2634

5. Initialization of the Elements F', h', C', S', L'

After the short period terms have been eliminated, the Hamiltonian becomes the function

$$\mathcal{H}' \equiv \mathcal{H}'(-,g',h',L',G',H')$$
$$= \mathcal{H}'_0 + J_2\mathcal{H}'_1 + \frac{1}{2} J_2^2\mathcal{H}'_2 + \frac{1}{6} J_2^3\mathcal{H}'_3 + \frac{1}{24} J_2^4\mathcal{H}'_4 \tag{9}$$

with

$$\mathcal{H}'_0 \equiv \mathcal{H}'_0(-,-,-,L',-,-) = -\frac{1}{2}\frac{\mu^2}{L'^2},$$

$$\mathcal{H}'_1 = \mathcal{H}'_1(-,-,-,L',G',H') = \mathcal{H}^1_0,$$

$$\mathcal{H}'_k = \mathcal{H}'_k(-,g',-,L',G',H') = \mathcal{H}^k_0, \qquad 2 \leq k \leq 4.$$

Now that ℓ' is ignorable in the transformed variable, the action L' turns out to be an integral.

On one hand, through the elimination of short period terms, the solution of the Main Problem reduces to integrating the Hamiltonian equations generated by (9) and composing their solutions with the equations of the canonical normalization. On the other hand, the initial conditions determine the initial values of the elements F, h, C, S, L and H, but not of the transformed elements. Therefore, to insure that the solution in (ℓ',g',h',L',G',H') corresponds through the transformation to the initial elements (F_0,h_0,C_0,S_0,L_0,H), we must provide the way of determining the initial values of the transformed elements $(F'_0,h'_0,C'_0,S'_0,L'_0,H)$. In a Calculus of Perturbations based on Lie transforms, the initialization of the transformed variables is accomplished by constructing explicitly the inverse mapping, or rather the generators (V_1,V_2,V_3) of the inverse mapping. As it was established elsewhere (Deprit 1969), at least up to the third order, they result from these straightforward operations:

$$V_1 \equiv V_1(\ell,g,-,L,G,H) = -W_1(\ell,g,-,L,G,H),$$
$$V_2 \equiv V_2(\ell,g,-,L,G,H) = -W_2(\ell,g,-,L,G,H),$$
$$V_3 \equiv V_3(\ell,g,-,L,G,H) = -W_3 - (W_2;W_1).$$

As we did in the case of the direct transformation, we by-passed the explicit construction of the inverse transformation itself, for the equations expressing ℓ' and g' in terms of ℓ, g, h, L, G, H will involve negative powers of e. Instead we addressed ourselves directly to the task of expressing the elements F', h', C', S', L' in terms of F, h, C, S, L. These expansions have the same form as the ones described in the previous section for the direct transformation. Inserting in these series the initial values F_0, h_0, C_0, S_0, L_0 and the value for the integral H would then provide the initial values F_0', h_0', C_0', S_0' and the value of the integral L'.

6. Elimination of the Long Period Terms

Following the main steps of Satellite Theory as outlined by Brouwer, we now consider constructing a completely canonical transformation from the variables (ℓ',g',h',L',G',H') into the variables $(\ell'',g'',h'',L'',G'',H'')$ to the effect of rendering the angle g'' ignorable in the transformed Hamiltonian.

The construction is again based on determining the transformation generators W_1, W_2, W_3, and so on. Thus entering a triangle similar to the one displayed at the beginning of Section 3, we put

$$\mathcal{H}_k^0(\ell'',g'',-,L'',G'',H'') = \mathcal{H}_k'(\ell'',g'',-,L'',G'',H'').$$

Order 1. The differential identity

$$\mathcal{H}_1^0 + (\mathcal{H}_0^0;\phi_1) = \mathcal{H}_0^1$$

is satisfied by putting

$$\mathcal{H}_0^1 = \mathcal{H}_1^0$$

and by requesting that

$$\partial\phi_1/\partial\ell'' = 0,$$

or that ϕ_1 do not depend on ℓ''.

According to what we observed in Section 3,

$$\mathcal{H}_0^1 = \mathcal{H}_0^1(-,-,-,L'',G'',H'')$$

$$= \mu^4 R_e^2 L''^{-6} \sum_{j\geq 0} e''^{2j} [\mathcal{H}^1_{0,j,0} + \mathcal{H}^1_{0,j,2}(H''/L'')^2].$$

Consequently

$$\frac{\partial \mathcal{H}_0^1}{\partial G''} = 2\mu^4 R_e^2 G'' L''^{-8} \sum_{j\geq 0} (j+1) e''^{2j} [\mathcal{H}^1_{0,j+1,0} + \mathcal{H}^1_{0,j+1,2}(H''/L'')^2]$$

$$= 2\mu^4 R_e^2 L''^{-7} (1-e''^2)^{\frac{1}{2}} \sum_{j\geq 0} e''^{2j} [\mathcal{H}^1_{0,j+1,0} + \mathcal{H}^1_{0,j+1,2}(H''/L'')^2].$$

The program has been set up to expand $(1-e'')^{\frac{1}{2}}$, to multiply this binomial development with the explicitly written power series in e''^2, and to bring $\partial\mathcal{H}_0^1/\partial G''$ in a power series of the form

$$\frac{\partial\mathcal{H}_0^1}{\partial G''} = D(L'',G'',H'') = 3/4\mu^4 R_e^2 L''^{-7}\Delta\left[1 + \sum_{j\geq 1}\Delta_j e''^j\right]$$

where

$$\Delta = 1 - 5\,\frac{H''^2}{L''^2}\,,$$

and, for any $j \geq 1$, the coefficient Δ_j is a polynomial in Δ and H''/L''. By introducing Δ as a symbol in the list of polynomial variables, it becomes feasible to expand 1/D in power series of e'' and, as a matter of fact, this artifice makes this inversion quite an elementary operation for an automatic processor of Poisson series like MAO. The power series 1/D plays an essential role at the subsequent orders in the elimination of the long period term.

<u>Order 2</u>. The element

$$\mathcal{H}_1^1 = \mathcal{H}_2^0 + (\mathcal{H}_1^0;\phi_1) + (\mathcal{H}_0^0;\phi_2)$$

reduces to

$$\mathcal{H}_1^1 = \mathcal{H}_2^0 + (\mathcal{H}_1^0;\phi_1)$$

if we assume that

$$\partial\phi_2/\partial\ell'' = 0,$$

i.e., if we select for the second order generator ϕ_2 a function that does not depend explicitly on ℓ''. Consequently

$$\mathcal{H}_0^2 = \mathcal{H}_1^1 + (\mathcal{H}_0^1;\phi_1) = \mathcal{H}_2^0 + 2(\mathcal{H}_1^0;\phi_1).$$

This relation constitutes a differential identity in the unknowns $\mathcal{H}_0^2$ and ϕ_1. We can satisfy by selecting for $\mathcal{H}_0^2$ the average of $\mathcal{H}_2^0$ over the angle g'', so that

$$\mathcal{H}_0^2 \equiv \mathcal{H}_0^2(-,-,-,L'',G'',H'') = \langle \mathcal{H}_2^0 \rangle_{g''},$$

by assuming that

$$\partial \phi_1 / \partial h'' = 0,$$

so that the identity reduces to the quadrature

$$\frac{\partial \phi_1}{\partial g''} = \frac{1}{2} \frac{1}{D} (\mathcal{H}_2^0 - \mathcal{H}_0^2).$$

The results of these operations are, on one hand for the transformed Hamiltonian, a second order component of the type

$$\mathcal{H}_0^2 = \mu^6 R_e^4 L''^{-10} \sum_{j \geq 0} e''^{2j} [\mathcal{H}_{0,j,0}^2 + \mathcal{H}_{0,j,2}^2 (H''/L'')^2 + \mathcal{H}_{0,j,4}^2 (H''/L'')^4]$$

and, on the other hand for the canonical transformation, a first order generator of the type

$$\phi_1 = \mu^2 R_e^2 L''^{-3} \sin 2g'' \sum_{j \geq 1} \Delta^{-j} e''^{2j} \sum_{0 \leq k \leq j+1} \phi_{1,j,k} (H''/L'')^{2k}.$$

<u>Order 3</u>. Entering the fourth row of the transformation triangle for $\mathcal{H}'$, we meet the element

$$\mathcal{H}_2^1 = \mathcal{H}_3^0 + (\mathcal{H}_2^0; \phi_1) + 2(\mathcal{H}_1^0; \phi_2) + (\mathcal{H}_0^0; \phi_3).$$

Assuming that

$$\partial \phi_3 / \partial \ell'' = 0,$$

we reduce it to the sum

$$\mathcal{H}_2^1 = \mathcal{H}_3^0 + (\mathcal{H}_2^0; \phi_1) + 2(\mathcal{H}_1^0; \phi_2).$$

We isolate in it the part yet unknown by putting

$$\tilde{\mathcal{H}}_2^1 = \mathcal{H}_3^0 + (\mathcal{H}_2^0;\phi_1)$$

so that

$$\mathcal{H}_2^1 = \tilde{\mathcal{H}}_2^1 + 2(\mathcal{H}_1^0;\phi_2).$$

The partial element $\tilde{\mathcal{H}}_2^1$ is to be computed whereas the Poisson bracket $(\mathcal{H}_1^0;\phi_2)$ is set aside until ϕ_2 has been determined.

We treat in the same way the elements

$$\mathcal{H}_1^2 = \mathcal{H}_2^1 + (\mathcal{H}_1^1;\phi_1) + (\mathcal{H}_0^1;\phi_2),$$
$$\mathcal{H}_0^3 = \mathcal{H}_1^2 + (\mathcal{H}_0^2;\phi_1).$$

By putting

$$\tilde{\mathcal{H}}_1^2 = \tilde{\mathcal{H}}_2^1 + (\mathcal{H}_1^1;\phi_1),$$
$$\tilde{\mathcal{H}}_0^3 = \tilde{\mathcal{H}}_1^2 + (\mathcal{H}_0^2;\phi_1),$$

so that

$$\mathcal{H}_1^2 = \tilde{\mathcal{H}}_1^2 + 3(\mathcal{H}_1^0;\phi_2),$$
$$\mathcal{H}_0^3 = \tilde{\mathcal{H}}_0^3 + 3(\mathcal{H}_1^0;\phi_2),$$

we isolate in them the parts that we are able to compute and isolate the Poisson bracket. In fact, the relation

$$\tilde{\mathcal{H}}_0^3 + 3(\mathcal{H}_1^0;\phi_2) = \mathcal{H}_0^3$$

constitutes a differential identity to be satisfied by the unknowns

$\mathcal{H}_0^3$ and Φ_2.

As we have done repeatedly, we propose to take for $\mathcal{H}_0^3$ the average of $\tilde{\mathcal{H}}_0^3$ over g'', and while we assume that

$$\partial\Phi_2/\partial h'' = 0,$$

we obtain Φ_2 from the quadrature

$$\frac{\partial\Phi_2}{\partial g''} = \frac{1}{3}\,\frac{1}{D}(\tilde{\mathcal{H}}_0^3 - \mathcal{H}_0^3).$$

These operations produce, on one hand for the averaged Hamiltonian, the third order component which is a series of the form

$$\mathcal{H}_0^3 = \mu^8 R_e^6 L''^{-14} \sum_{j\geq 0} \Delta^{-j} e''^{2j} \sum_{0\leq k\leq j+3} \mathcal{H}^3_{0,j,k} (H''/L'')^{2k},$$

and, on the other hand for the canonical transformation, the second order generator

$$\Phi_2 = \mu^4 R_e^4 L''^{-7} \sum_{j\geq 1} \Delta^{-j-1} e''^{2j} \sum_{0\leq k\leq j+3} \Phi_{2,j,k} (H''/L'')^{2k},$$

the coefficients $\Phi_{2,j,k}$ being finite sums of sine functions in the arguments $2pg''$ with $1 \leq p \leq \mathrm{Min}(j,2)$.

Order 4. Before computing the fourth order row in the triangle, we complete the elements of the third order:

$$\mathcal{H}_2^1 = \tilde{\mathcal{H}}_2^1 - \frac{2}{3}(\tilde{\mathcal{H}}_0^3 - \mathcal{H}_0^3).$$

$$\mathcal{H}_1^2 = \tilde{\mathcal{H}}_1^2 - (\tilde{\mathcal{H}}_0^3 - \mathcal{H}_0^3).$$

Then in the element

$$\mathcal{H}_3^1 = \mathcal{H}_4^0 + (\mathcal{H}_3^0;\phi_1) + 3(\mathcal{H}_2^0;\phi_2) + 3(\mathcal{H}_1^0;\phi_3) + (\mathcal{H}_0^0;\phi_4),$$

we assume that

$$\partial\phi_4/\partial\ell'' = 0,$$

we isolate the terms that we can already compute at this stage, namely

$$\tilde{\mathcal{H}}_3^1 = \mathcal{H}_4^0 + (\mathcal{H}_3^0;\phi_1) + 3(\mathcal{H}_2^0;\phi_2)$$

so that

$$\mathcal{H}_3^1 = \tilde{\mathcal{H}}_3^1 + 3(\mathcal{H}_1^0;\phi_3).$$

Similarly in the element

$$\mathcal{H}_2^2 = \mathcal{H}_3^1 + (\mathcal{H}_2^1;\phi_1) + 2(\mathcal{H}_1^1;\phi_2) + (\mathcal{H}_0^1;\phi_3)$$

we group the terms that we are already able to evaluate:

$$\tilde{\mathcal{H}}_2^2 = \tilde{\mathcal{H}}_3^1 + (\mathcal{H}_2^1;\phi_1) + 2(\mathcal{H}_1^1;\phi_2)$$

and this way we come to the relation

$$\mathcal{H}_2^2 = \tilde{\mathcal{H}}_3^1 + 4(\mathcal{H}_1^0;\phi_3).$$

Proceeding likewise for the element

$$\mathcal{H}_1^3 = \mathcal{H}_2^2 + (\mathcal{H}_1^2;\phi_1) + (\mathcal{H}_0^2;\phi_2),$$

we put

$$\tilde{\mathcal{H}}_1^3 = \tilde{\mathcal{H}}_2^2 + (\mathcal{H}_1^2;\phi_1) + (\mathcal{H}_0^2;\phi_2)$$

and arrive at the decomposition

$$\mathcal{H}_1^3 = \tilde{\mathcal{H}}_1^3 + 4(\mathcal{H}_1^0;\phi_3).$$

Finally, by introducing

$$\tilde{\mathcal{H}}_0^4 = \tilde{\mathcal{H}}_1^3 + (\mathcal{H}_0^3;\phi_1),$$

we are led to the differential identity

$$\tilde{\mathcal{H}}_0^4 + 4(\mathcal{H}_1^0;\phi_3) = \mathcal{H}_0^4.$$

Hence, after transfering into $\mathcal{H}_0^4$ the terms of $\tilde{\mathcal{H}}_0^4$ that do not depend on g'', we perform the quadrature

$$\frac{\partial\phi_3}{\partial g''} = \frac{1}{4}\,\frac{1}{D}(\tilde{\mathcal{H}}_0^4 - \mathcal{H}_0^4)$$

to obtain the third order generator of the canonical transformation. This is a power series in e''^2 of the form

$$\phi_3 = \mu^6 R_e^6 L''^{-11} \sum_{j\geq 1} \Delta^{-j-2} e''^{2j} \sum_{0\leq k\leq j+4} \phi_{3,j,k} (H''/L'')^{2k}$$

whose coefficients $\phi_{3,j,k}$ are finite sums of sine functions in the arguments $2pg''$ with $1 \leq p \leq \mathrm{Min}(j,3)$.

We enlist in Table VII the number of terms contained in the result series obtained so far. This summary indicates how the elimination of short period terms has simplified to a very large extent the further reduction of the Main Problem. But the reduction has been bought at the expense of more sophisticated manipulations in the construction of the now completely normalized Hamiltonian.

Table VII. Recapitulation of the Hamiltonian averaged over the long period angle and of the generators of the averaging transformation

	$\mathcal{H}_0^1$	Φ_1	$\mathcal{H}_0^2$	Φ_2	$\mathcal{H}_0^3$	Φ_3	$\mathcal{H}_0^4$
e''^0	2	0	3	0	4	0	5
e''^2	2	3	3	5	5	7	7
e''^4	2	4	3	12	6	16	8
e''^6	2	5	3	14	7	27	9
e''^8	2	6	3	16	8	30	10
e''^{10}	2	7	3	18	9	33	11
e''^{12}	2	8	3	20	10		
e''^{14}	2	9	3				
e''^{16}	2						
Total	18	42	24	85	49	113	50

7. The Elements of the Long Period Elimination and Their Initialization

The generators of the transformation from (ℓ',g',h',L',G',H') to $(\ell'',g'',h'',L'',G'',H'')$ do not depend on the angles ℓ'' and h''; hence the transformation does not change the actions L' and H': thus

$$L' = L'' \quad \text{and} \quad H' = H''.$$

In order to avoid the presence of divisors in e'', we do not determine the transformation through its equations expressing ℓ', g', h'

and G' in terms of the new variables, but rather by means of the elements $F' = \ell' + g'$, h', $C' = e'\cos g'$ and $S' = e'\sin g'$. The algorithm by which these state variables are expressed in terms of the new variables with the help of the generators ϕ_1, ϕ_2, ϕ_3 has been described elsewhere (Deprit 1969). Here it suffices to say to record the morphology of the resulting series. For the sake of brevity, we have put

$$\eta'' = H''/L'', \qquad \alpha'' = a''/R_e.$$

a) <u>The mean distance F' to the ascending node</u>

$$F' = F'' + J_2\alpha''^{-2}F''_1 + \frac{1}{2} J_2^2\alpha''^{-4}F''_2 + \frac{1}{6} J_2^3\alpha''^{-6}F''_3,$$

$$F''_1 = \sum_{1\le j\le 6} e''^{2j}\Delta^{-j-1} \sum_{0\le k\le j+2} F''_{1,j,k}\eta''^{2k},$$

$$F''_2 = \sum_{1\le j\le 5} e''^{2j}\Delta^{-j-2} \sum_{0\le k\le j+4} F''_{2,j,k}\eta''^{2k},$$

$$F''_3 = \sum_{1\le j\le 4} e''^{2j}\Delta^{-j-3} \sum_{0\le k\le j+6} F''_{3,j,k}\eta''^{2k}.$$

The coefficients $F''_{r,j,k}$ are sums of sine functions in the arguments $2pg''$ with $1 \le p \le \mathrm{Min}(r,j)$.

b) <u>The longitude h' of the ascending node</u>

$$h' = h'' + \eta''(J_2\alpha''^{-2}h''_1 + \frac{1}{2} J_2^2\alpha''^{-4}h''_2 + \frac{1}{6} J_2^3\alpha''^{-6}h''_3),$$

$$h''_1 = \sum_{1\le j\le 7} e''^{2j}\Delta^{-j-1} \sum_{0\le k\le j+1} h''_{1,j,k}\eta''^{2k},$$

$$h''_2 = \sum_{1\le j\le 6} e''^{2j}\Delta^{-j-2} \sum_{0\le k\le j+3} h''_{2,j,k}\eta''^{2k},$$

$$h''_3 = \sum_{0\le j\le 5} e''^{2j}\Delta^{-j-3} \sum_{0\le k\le j+5} h''_{3,j,k}\eta''^{2k}.$$

The coefficients $h'''_{r,j,k}$ are sums of sine functions in the arguments $2pg''$ with $1 \le p \le \mathrm{Min}(r,j)$.

c) <u>The scalar $C' = e'\cos g'$</u>

$$C' = C'' + e''(J_2\alpha''^{-2}C''_1 + \frac{1}{2} J_2^2\alpha''^{-4}C''_2 + \frac{1}{6} J_2^3\alpha''^{-6}C''_3),$$

$$C''_1 = \sum_{0\le j\le 6} e''^{2j}\Delta^{-j-1} \sum_{0\le k\le j+2} C''_{1,j,k}\eta''^{2k},$$

$$C''_2 = \sum_{0\le j\le 5} e''^{2j}\Delta^{-j-2} \sum_{0\le k\le j+4} C''_{2,j,k}\eta''^{2k},$$

$$C''_3 = \sum_{0\le j\le 4} e''^{2j}\Delta^{-j-3} \sum_{0\le k\le j+5} C''_{3,j,k}\eta''^{2k}.$$

The coefficients $C''_{r,j,k}$ are sums of cosine functions in the arguments $(2p+1)g''$ with $1 \le p \le \mathrm{Min}(r,j)$.

d) <u>The scalar $S' = e'\sin g'$</u>

$$S' = S'' + e''(J_2\alpha''^{-2}S''_1 + \frac{1}{2} J_2^2\alpha''^{-4}S''_2 + \frac{1}{6} J_2^3\alpha''^{-6}S''_3),$$

$$S''_1 = \sum_{0\le j\le 6} e''^{2j}\Delta^{-j-1} \sum_{0\le k\le j+2} S''_{1,j,k}\eta''^{2k},$$

$$S''_2 = \sum_{0\le j\le 5} e''^{2j}\Delta^{-j-2} \sum_{0\le k\le j+4} S''_{2,j,k}\eta''^{2k},$$

$$S''_3 = \sum_{0\le j\le 4} e''^{2j}\Delta^{-j-3} \sum_{0\le k\le j+6} S''_{3,j,k}\eta''^{2k}.$$

The coefficients $S''_{r,j,k}$ are sums of sine functions in the arguments $(2p+1)g''$ with $1 \le p \le \mathrm{Min}(r,j)$.

The inverse transformation, i.e., the mapping from $(\ell'',g'',h'',L'',G'',H'')$

Table VIII. Summary of the long period corrections to F', h', e'cos g' and e'sin g'

$J_2 \varepsilon''^{-2}$	e''	ΔF''	Δh''	ΔC''	ΔS''
1	0	0	0	3	3
	2	4	3	8	8
	4	5	4	10	10
	6	6	5	12	12
	8	7	6	14	14
	10	8	7	16	16
	12	9	8	18	18
	14		9		
		39	42	81	81
2	0	0	0	5	5
	2	6	5	12	12
	4	14	12	21	21
	6	16	14	24	24
	8	18	16	27	27
	10	20	18	30	30
	12		20		
		74	85	119	119
3	0	0	0	7	7
	2	8	7	16	16
	4	18	16	27	27
	6	30	27	40	40
	8	33	30	44	44
	10		33		
		89	113	134	134
	Total	202	240	334	334

to (ℓ',g',h',L',G',H') is generated from functions $\psi_1, \psi_2, \psi_3, \ldots,$ which are easily derived from the generators $\phi_1, \phi_2, \phi_3, \ldots$ of the direct transformation:

$$\psi_1(-,g',-,L',G',H') = -\phi_1(-,g',-,L',G',H'),$$
$$\psi_2(-,g',-,L',G',H') = -\phi_2(-,g',-,L',G',H'),$$
$$\psi_3(-,g',-,L',G',H') = -\phi_3 - (\phi_2;\phi_1).$$

These generators will be used to express the state variables F'', h'', C'', S'' in terms of F', h', C', S', L' and H'. In particular, if we give to the elements the initial values F_0', h_0', C_0', S_0', L' that we computed in Section 5, we obtain from the series the corresponding initial values F_0'', h_0'', C_0'' and S_0'' of the elements resulting from the elimination of the long period terms.

8. The Secular Terms

After elimination of the long period terms, the Hamiltonian of the Main Problem is the series

$$\begin{aligned}\mathcal{H}'' &= \mathcal{H}''(-,-,-,L'',G'',H'') \\ &= \mathcal{H}_0'' + J_2\mathcal{H}_1'' + \frac{1}{2} J_2^2\mathcal{H}_2'' + \frac{1}{6} J_2^3\mathcal{H}_3''\end{aligned}$$

with

$$\mathcal{H}_0'' \equiv \mathcal{H}_0''(-,-,-,L'',-,-) = -\frac{1}{2}\frac{\mu^2}{L''^2},$$

$$\mathcal{H}_1'' \equiv \mathcal{H}_1''(-,-,-,L'',G'',H'') = \mathcal{H}_0^1,$$

$$\mathcal{H}_2'' \equiv \mathcal{H}_2''(-,-,-,L'',G'',H'') = \mathcal{H}_0^2.$$

The canonical equations it generates have a trivial solution. Since the angle coordinates are all ignorable, the actions L", G" and H" are constants of the motion. Therefore, the frequencies

$$\nu_1 = \frac{\partial\mathcal{H}''}{\partial L''}, \qquad \nu_2 = \frac{\partial\mathcal{H}''}{\partial G''}, \qquad \nu_3 = \frac{\partial\mathcal{H}''}{\partial H''}$$

are also constants. Let us review their morphology:

a) <u>Anomalistic mean motion</u> ν_1

$$\nu_1 = n''(1+J_2\alpha''^{-2}\nu_{1,1}+\frac{1}{2}J_2^2\alpha''^{-4}\nu_{1,2}+\frac{1}{6}J_2^3\alpha''^{-6}\nu_{1,3}),$$

$$\nu_{1,1} = \sum_{0\le j\le 7} e''^{2j}(\nu_{1,1,j,0}+\nu_{1,1,j,2}\eta''^2),$$

$$\nu_{1,2} = \sum_{0\le j\le 6} e''^{2j}(\nu_{1,2,j,0}+\nu_{1,2,j,2}\eta''^2+\nu_{1,2,j,4}\eta''^4),$$

$$\nu_{1,3} = \sum_{0\le j\le 5} e''^{2j}\Delta^{-j-1}\sum_{0\le k\le j+4}\nu_{1,3,j,k}\eta''^{2k}.$$

b) <u>Mean motion of the perigee</u> ν_2

$$\nu_2 = J_2 n''\alpha''^{-2}(\nu_{2,1}+\frac{1}{2}J_2\alpha''^{-2}\nu_{2,2}+\frac{1}{6}J_2^2\alpha''^{-2}\nu_{2,3}),$$

$$\nu_{2,1} = \sum_{0\le j\le 7} e''^{2j}(\nu_{2,1,j,0}+\nu_{2,1,j,2}\eta''^2),$$

$$\nu_{2,2} = \sum_{0\le j\le 6} e''^{2j}(\nu_{2,2,j,0}+\nu_{2,2,j,2}\eta''^2+\nu_{2,2,j,4}\eta''^4),$$

$$\nu_{2,3} = \sum_{0\le j\le 5} e''^{2j}\Delta^{-j-1}\sum_{0\le k\le j+4}\nu_{2,3,j,k}\eta''^{2k}.$$

c) <u>Mean motion of the ascending node</u> ν_3

$$\nu_3 = J_2 n''\alpha''^{-2}\eta''(\nu_{3,1}+\frac{1}{2}J_2\alpha''^{-2}\nu_{3,2}+\frac{1}{6}J_2^2\alpha''^{-4}\nu_{3,3}),$$

$$\nu_{3,1} = \sum_{0\le j\le 8}\nu_{3,1,j}e''^{2j},$$

$$\nu_{3,2} = \sum_{0\le j\le 7} e''^{2j}(\nu_{3,2,j,0}+\nu_{3,2,j,2}\eta''^2),$$

$$\nu_{3,3} = \sum_{0\le j\le 6} e''^{2j}\Delta^{-j-1} \sum_{0\le k\le j+3} \nu_{3,3,j,k}\eta''^{2k}.$$

Instead of integrating the differential equations in ℓ'' and g'', which raises for the initial conditions ℓ''_0 and g''_0 the problem of divisions by the small eccentricities e_0 and e'_0, we determine the elements F'', C'', S''. Thus the integration of the secular part of the Main Problem results in the following formulas:

$$F'' = \ell'' + g'' = (\nu_1+\nu_2)(t-t_0) + F''_0,$$

$$C'' = e''\cos g'' = C''_0\cos \nu_2(t-t_0) - S''_0\sin \nu_2(t-t_0),$$

$$S'' = e''\sin g'' = S''_0\cos \nu_2(t-t_0) + C''_0\sin \nu_2(t-t_0),$$

$$h'' = \nu_3(t-t_0) + h''_0.$$

9. <u>Computation of the Coordinates and Velocities</u>

After the series for the direct and inverse transformations eliminating all periodic terms have been obtained, we are in a position to compute at any instant--within the time interval of validity of the theory--the position and the velocity of the satellite from coordinates and velocity components at an epoch t_0.

Given $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0)$, the initial values of the elements F_0, h_0, C_0, S_0, L_0, H are determined from the following system of formulas:

The components of the angular momentum being

$$A_0 = y_0\dot{z}_0 - z_0\dot{y}_0,$$

$$B_0 = z_0\dot{x}_0 - x_0\dot{z}_0,$$

$$H = x_0\dot{y}_0 - y_0\dot{x}_0,$$

we compute its norm at t_0,

$$G_0 = \sqrt{A_0^2 + B_0^2 + H^2},$$

so that the trigonometric relations

$$A_0 = G_0 \sin I_0 \sin h_0,$$

$$B_0 = -G_0 \sin I_0 \cos h_0,$$

$$H = G_0 \cos I_0$$

determine unambiguously the initial longitude h_0 of the ascending node and the initial inclination. Next we rotate the coordinate axes so that $O\xi$ and $O\eta$ lie in the orbital plane $\Pi(t_0)$ with $O\xi$ coinciding with the ascending node at the initial instant t_0. In this frame of reference, the initial coordinates and velocities of the satellite are

$$\xi_0 = x_0 \cos h_0 + y_0 \sin h_0,$$

$$\eta_0 = (-x_0 \sin h_0 + y_0 \cos h_0) \cos I_0 + z_0 \sin I_0,$$

$$\dot{\xi}_0 = \dot{x}_0 \cos h_0 + \dot{y}_0 \sin h_0,$$

$$\dot{\eta}_0 = (-\dot{x}_0 \sin h_0 + \dot{y}_0 \cos h_0) \cos I_0 + \dot{z}_0 \sin I_0.$$

Evidently the initial planetocentric distance is

$$r_0 = \sqrt{\xi_0^2 + \eta_0^2}.$$

Moreover, from Laplace-Hamilton's vector oriented along the apsidal line, we obtain that

$$C_0 = G_0 \dot{\eta}_0 - \xi_0 r_0^{-1}$$

$$S_0 = -G_0 \dot{\xi}_0 - \eta_0 r_0^{-1},$$

which relations determine the square of the eccentricity

$$e_0^2 = C_0^2 + S_0^2.$$

Having evaluated the quantity

$$2E_0 = (\dot{\xi}_0^2+\dot{\eta}_0^2) - 2r_0^{-1},$$

we are now in a position to calculate the initial semi-major axis

$$a_0 = -1/2E_0,$$

hence Delaunay's action at time t_0, namely

$$L_0 = \sqrt{a_0}.$$

There remains to compute F_0. Recalling that

$$\frac{\xi_0}{a} = \cos\psi_0 - C_0 + \frac{S_0}{1+\sqrt{1-e_0^2}}(C_0\sin\psi_0 - S_0\cos\psi_0),$$

$$\frac{\eta_0}{a} = \sin\psi_0 - S_0 - \frac{C_0}{1+\sqrt{1-e_0^2}}(C_0\sin\psi_0 - S_0\cos\psi_0),$$

where $\psi_0 = E_0 + g_0$, (E_0 being the eccentric anomaly at time t_0), we deduce that

$$\frac{C_0\sin\psi_0 - S_0\cos\psi_0}{1+\sqrt{1-e_0^2}} = \frac{C_0\eta_0 - S_0\xi_0}{a_0\sqrt{1-e_0^2}},$$

so that the preceding formulas yield

$$\cos\psi_0 = C_0e_0^2 + \frac{\xi_0}{a_0} + \frac{S_0(C_0\eta_0 - S_0\xi_0)}{a_0}\,\frac{1-\sqrt{1-e_0^2}}{\sqrt{1-e_0^2}}$$

$$\sin\psi_0 = S_0e_0^2 + \frac{\eta_0}{a_0} - \frac{C_0(C_0\eta_0 - S_0\xi_0)}{a_0}\,\frac{1-\sqrt{1-e_0^2}}{\sqrt{1-e_0^2}}.$$

They determine unambiguously ψ_0. Finally an obvious modification of Kepler's equation produces

$$F_0 = \psi_0 - C_0\sin\psi_0 + S_0\cos\psi_0.$$

The initial values F_0, h_0, C_0, S_0, L_0, H are entered into the series expressing the averaged elements F', h', C', S', L' in terms of the osculating elements F, h, C, S, L and H; the substitution produces the initial values F_0', h_0', C_0', S_0' and the value of the integral L'. They are then introduced in the series expressing F'', h'', C'' and S'' in terms of F', h', C', S', L' and H, which results in the determination of the initial values F_0'', h_0'', C_0'', and S_0''.

At this stage we are in a position to evaluate from the series the numerical values of the three basic mean motions ν_1, ν_2 and ν_3. This is the last step in the initialization phase of the ephemeris.

The calculation of the position and velocity at any instant t follows the same line as the initialization, but in the reverse order.

First we evaluate the state variables F'', h'', C'' and S'' at time t from the simple formulas given at the end of Section 8. Then the series described in Section 7 provide the average state variables F', h', C', S', L' at time t, whereas the series of Section 4 furnish the osculating elements F, h, C, S, L at that instant.

After the process of evaluating the series numerically is completed, we determine the coordinates and components of the velocity from the following system of formulas:

Kepler's equation in the form

$$\psi = F + C \sin \psi - S \cos \psi$$

is solved by iteration to obtain the anomaly $\psi = E + g$. The eccentricity and the semi-major axis are computed from the relations

$$e^2 = C^2 + S^2, \qquad a = L^2.$$

The position in the nodal frame of reference is given by

$$\xi = a\left[\cos\psi - C + \frac{S}{1+\sqrt{1-e^2}}(C\sin\psi - S\cos\psi)\right],$$

$$\eta = a\left[\sin\psi - S - \frac{C}{1+\sqrt{1-e^2}}(C\sin\psi - S\cos\psi)\right].$$

Knowing

$$G = L\sqrt{1-e^2}, \qquad r = \sqrt{\xi^2+\eta^2},$$

we can use Hamilton's vector to get the components of the velocity in the nodal frame,

$$\dot\xi = -\frac{1}{G}\left(S + \frac{\eta}{r}\right),$$

$$\dot\eta = \frac{1}{G}\left(C + \frac{\xi}{r}\right).$$

Thereafter, having produced the inclination from the relations

$$\cos I = H/G, \qquad 0 \le I \le \pi,$$

we perform the usual rotations to pass from the nodal frame to the original inertial frame, and we come finally to the Cartesian coordinates

$$x = \xi\cos h - \eta\cos I\sin h,$$

$$y = \xi\sin h + \eta\cos I\cos h,$$

$$z = \eta\sin I,$$

and the components of the velocity

$$\dot x = \dot\xi\cos h - \dot\eta\cos I\sin h,$$

$$\dot y = \dot\xi\sin h + \dot\eta\cos I\cos h,$$

$$\dot z = \dot\eta\sin I$$

10. Reliability Tests

While the coding was in development, repeated checks were applied to test its correctness. The series generated by the program were constantly checked for their physical dimensions and their d'Alembert characteristic. Subroutines that, in the parlance of the trade, are called *expert*, like taking partial derivatives with respect to Delaunay's actions G and L or computing a Poisson bracket, have been thoroughly tested. For instance, in regard to the canonical transformation $(\ell,g,h,L,G,H) \rightarrow (\ell',g',h',L',G',H')$, we satisfied ourselves that the Poisson bracket $(\ell;g)$ is indeed equal to zero, i.e., that the coefficients in the result are smaller than 10^{-12} in relative accuracy. (N.B. They should have come exactly equal to zero had we operated in integer arithmetic.) The general course of a reduction by Lie transforms has been implemented on a number of simple examples like the simple pendulum, Duffing's equation and the relativistic harmonic oscillator.

The ultimate test of reliability is a comparison of the positions and velocities predicted by the series with those of a highly accurate numerical integration scheme.

We chose to integrate the equations of the Main Problem by recurrent power series. All Taylor expansions involved are computed at each step through degree 16; the time step is selected so as to maintain 12 significant figures in both the integral of energy and the integral of polar angular momentum. The procedure is rapid, highly accurate, and stable; it enables one to follow for very long arcs the orbit corresponding to a given set of initial conditions (Deprit and Zahar 1966).

In order to allow comparisons with other prediction methods, we borrowed from Bonavito *et al* (1968) the initial Cartesian coordinates and velocities they assign for the artificial satellites RELAY II and ANNA 1B. Table IX lists the initial conditions, the osculating elements at epoch, the corrections to be applied in order to obtain from them the constants of the motion as per our theory, and the basic periods.

We examined the residuals (integration-series) for the elliptic elements selected in our theory. Disagreements on the inclination I and in the longitude of the node h are simply insignificant. As expected, the mean distance F to the node shows a secular deviation (see Fig. 1). For ANNA 1B, the residuals in the semi-major axis suggest a long period error coupled with a secular trend; for RELAY II, the presence of a long period error is well marked in the element C. For all other variables presented in Figure 1, the effects of short period errors caused by the truncatures in the eccentricity and J_2 seem to mask long period and secular tendencies.

More revealing quantities to evaluate long range reliability are the *intrinsic* deviations. If x_I, y_I, z_I (resp. x_S, y_S, z_S) are the coordinates at time t furnished by the numerical integration (resp. the literal series), and if X_I, Y_I, Z_I (resp. X_S, Y_S, Z_S) are the components of the velocities, the directions of the tangent, binormal and normal to the orbit at time t are given by the triplets

Table IX. Constants of the motion for the test orbits

	ANNA 1B	RELAY II
Initial osculating elements		
F_0 (in radians)	2.538 875 214 278	3.273 083 992 516
h_0 (in radians)	0.949 636 751 294	-2.384 959 105 384
S_0	-0.002 107 639 831	-0.025 229 668 345
C_0	-0.006 371 881 838	-0.234 623 580 641
L_0 (in Vanguard units)	1.085 131 662 111	1.322 050 356 567
H (in Vanguard units)	0.695 348 576 283	0.884 318 864 870
Short period corrections		
$F_0' - F_0$	$0.273\ 044\ 549 \times 10^{-3}$	$-0.052\ 347\ 711 \times 10^{-3}$
$h_0' - h_0$	$0.342\ 375\ 395 \times 10^{-3}$	$-0.006\ 726\ 021 \times 10^{-3}$
$S_0' - S_0$	$-0.369\ 163\ 708 \times 10^{-3}$	$0.123\ 600\ 234 \times 10^{-3}$
$C_0' - C_0$	$0.015\ 809\ 392 \times 10^{-3}$	$0.563\ 272\ 260 \times 10^{-3}$
$L' - L_0$	$-0.128\ 216\ 782 \times 10^{-3}$	$-0.452\ 874\ 015 \times 10^{-3}$
Long period corrections		
$F_0'' - F_0'$	$0.005\ 752 \times 10^{-6}$	$0.846\ 017 \times 10^{-6}$
$h_0'' - h_0'$	$0.012\ 755 \times 10^{-6}$	$2.070\ 715 \times 10^{-6}$
$S_0'' - S_0'$	$-0.561\ 488 \times 10^{-6}$	$-2.404\ 673 \times 10^{-6}$
$C_0'' - C_0'$	$1.410\ 317 \times 10^{-6}$	$21.619\ 075 \times 10^{-6}$
Periods		
$2\pi/\nu_1$	$1^h47^m10^s.2139$	$3^h15^m00^s.4528$
$2\pi/\nu_2$	$121^d05^h38^m15^s.35$	$331^d03^h50^m54^s.43$
$2\pi/\lvert\nu_3\rvert$	$99^d17^h34^m55^s.60$	$332^d22^h12^m57^s.11$

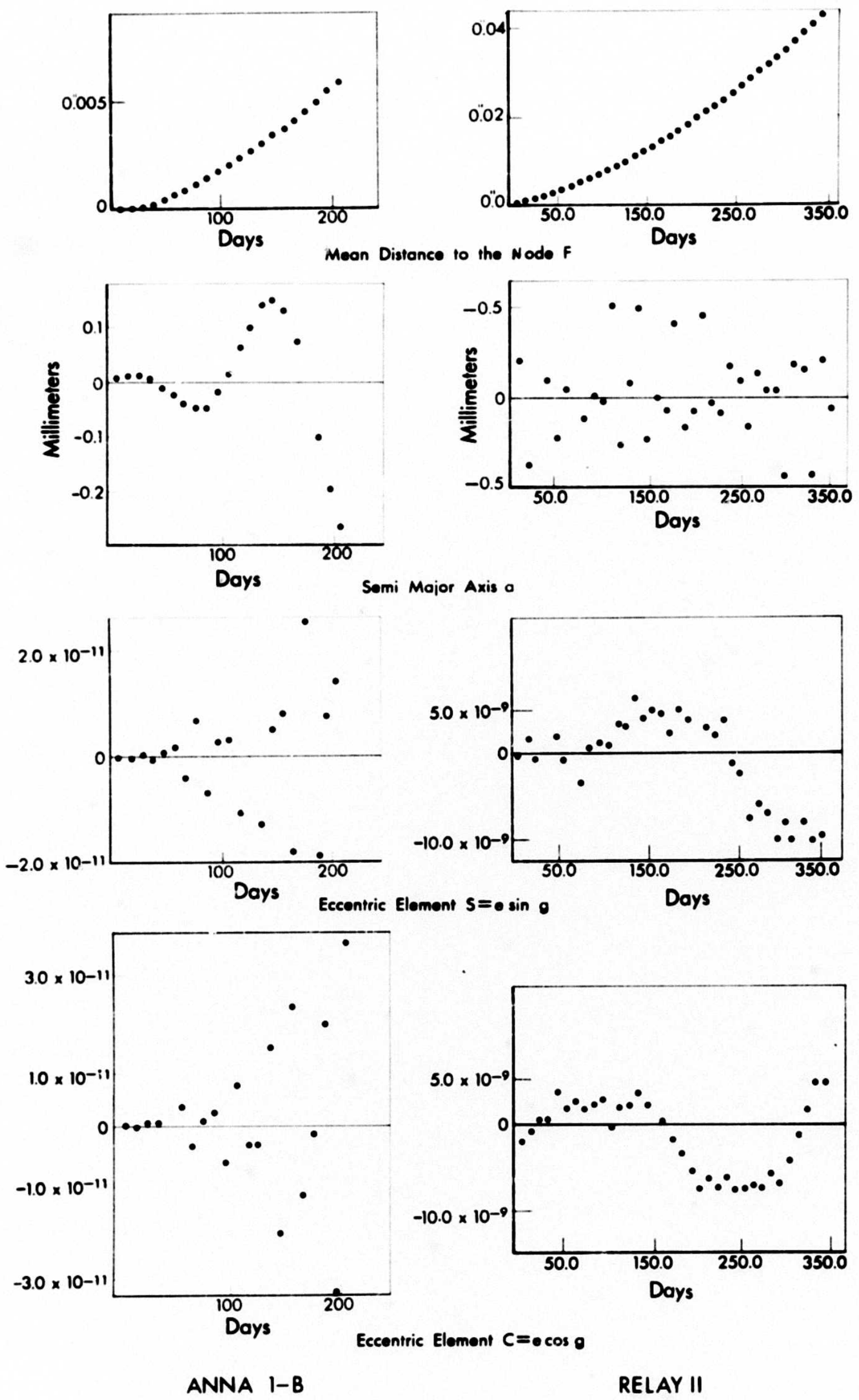

Fig. 1. Errors on osculating elliptic elements

$$\underset{\sim}{t}_I = (X_I/V_I, Y_I/V_I, Z_I/V_I)$$

where

$$V_I = (X_I^2+Y_I^2+Z_I^2)^{\frac{1}{2}},$$

$$\underset{\sim}{b}_I = (A_I/G_I, B_I/G_I, C_I/G_I)$$

where

$$A_I = y_I Z_I - z_I Y_I,$$
$$B_I = z_I X_I - x_I Z_I,$$
$$C_I = x_I Y_I - y_I X_I,$$
$$G_I = (A_I^2+B_I^2+C_I^2)^{\frac{1}{2}},$$

and

$$\underset{\sim}{n}_I = \Big((B_I Z_I - C_I Y_I)/G_I V_I, (C_I X_I - A_I Z_I)/G_I V_I, (A_I Y_I - B_I X_I)/G_I V_I\Big).$$

Then the dot products

$$(\underset{\sim}{x}_I - \underset{\sim}{x}_S) \cdot \underset{\sim}{t}_I,\ (\underset{\sim}{x}_I - \underset{\sim}{x}_S) \cdot \underset{\sim}{n}_I,\ (\underset{\sim}{x}_I - \underset{\sim}{x}_S) \cdot \underset{\sim}{b}_I$$

constitute the projections of the error in position respectively on the tangent ("in-track error"), the normal ("along-track error") and the binormal ("across-track error") of the orbit at time t. It is a characteristic feature of a perturbation theory that, while it yields a very close approximation of the orbit even over very long range, it leaves in error the basic clocks of the motion, namely the frequencies ν_1, ν_2, ν_3. However small the level of errors on these clocks, they cumulate linearly with time. The predicted orbit coincides nicely with the real trajectory of its initial conditions, but the satellite takes a more and more pronounced habit of arriving too early or too late at the expected meeting places. This story can be read in the diagrams of position errors (Figure 2). The deviations along and across the track are totally insignificant, but the errors in track show a secular trend. (Note that for both satellites, the tests cover more than one revolution of the perigee.) Comparison with similar tests by Arsenault *et al* (1963), Lubowe (1966) and Bonavito *et al* (1969) should restore confidence in the capabilities of a satellite theory based on Delaunay's elements. For ANNA 1B, after more than 3000 revolutions, the in-track error reaches only 0.2 meter; as for RELAY II, after 2800 revolutions, it is still as small as 2 meters.

Conclusions

In Perturbation Theory, Lie transforms will likely supersede Von Zeipel's method; they provide easy routine schemes for inverting canonical transformations, for determining the constants of the motion, and for transposing state variables.

Analytical expansions, however large the number of terms in the series, are capable of very high accuracy over long intervals of time. As a matter of fact, a Satellite Theory carried through the fourth order in its secular terms can deliver the accuracy presently contemplated by radar and laser experiments.

Computers can be programmed to generate developments of average complexity.

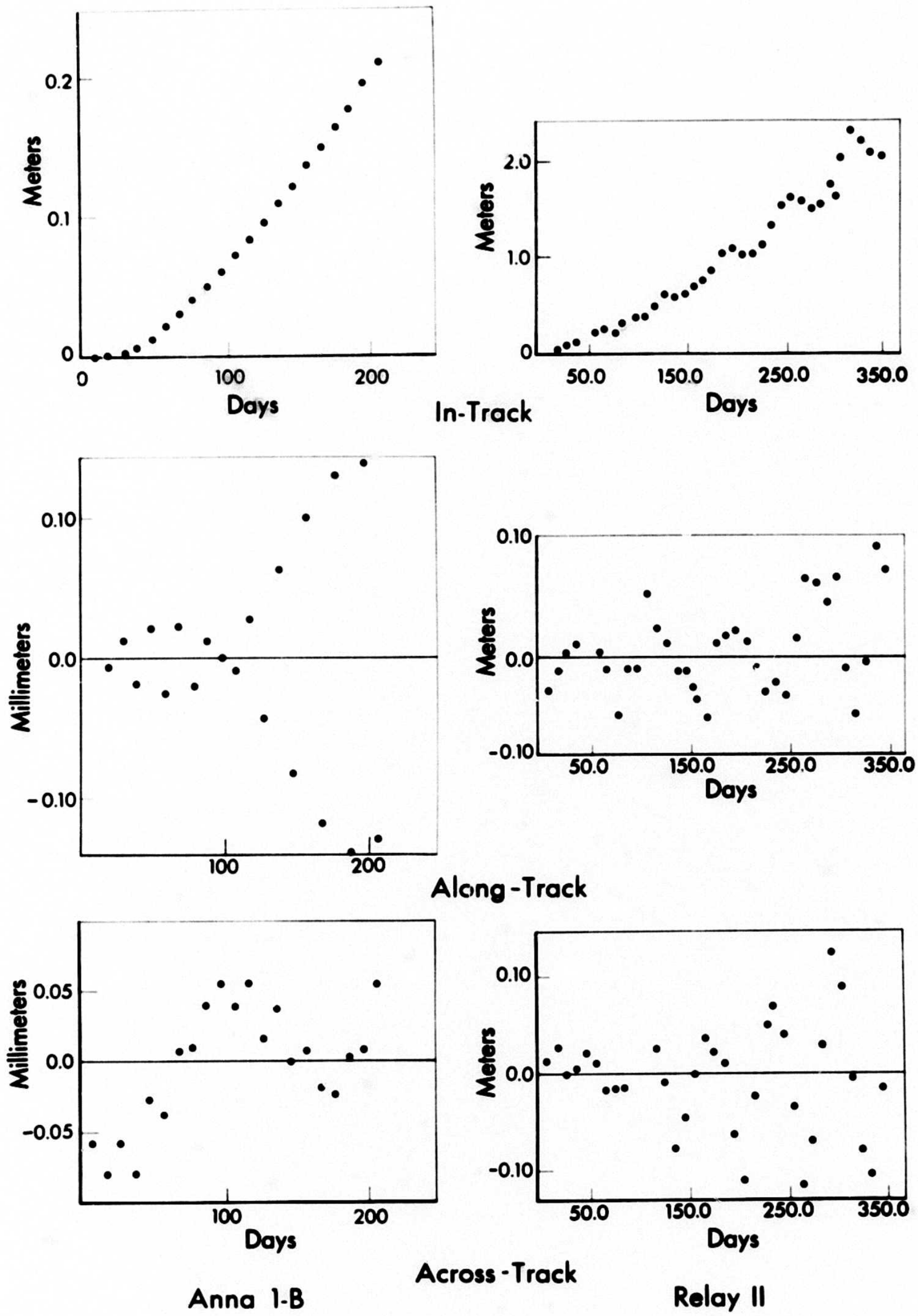

Fig. 2. Intrinsic errors in position (integration-series)

The present theory eradicates the difficulties caused by small eccentricities. It is evidently incomplete. But it has solved in principle the problems to be encountered from including more terms of the gravitational field, the luni-solar perturbations and other perturbations derived from force functions.

Readers interested in having the series produced by the present algorithm should contact the first author of the paper.

Acknowledgements

The outline of the present paper has been presented at the VII Annual Seminar in Celestial Mechanics, The University of Texas at Austin, January 13-14, 1969, where Dr. Boris Garfinkel gave us the opportunity of discussing in detail some of its aspects. Then followed a research report at the 50th Meeting of the American Geophysical Union, Washington, April 24, 1969. Once the research was completed, it became the topic of our lectures at the Summer Institute in Dynamical Astronomy, Massachusetts Institute of Technology, June 30-July 2, 1969. We are grateful to Dr. Musen and Dr. Vinti for their comments and suggestions at that occasion.

The IBM 360-44 of our laboratories is not suited to perform in one stretch the very long operations involved in the present theory, particularly those commanded by the elimination of the short period terms. In this respect we are especially grateful to our operator, Mr. John [illegible]; his sense of humor and his patient enthusiasm eventually overcame the limitations and failures of our equipment.

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